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Differentiable Manifolds

1 Definition of Manifold

$\mathbb{R}^n$ is an $n$-dim Cartesian or vector or linear space.

**Definition.** A *region*, or a region without boundary, ("open set") is a set $D$ of points in $\mathbb{R}^n$ such that together with each point $P_0$, $D$ also contains all points sufficiently close to $P_0$, i.e.

$$\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 :$$

all points $P = (x^1, \ldots, x^n)$ satisfying $|x^i - x_0^i| < \epsilon, i = 1, \ldots, n$ also lie in $D$. 
\( \mathbb{R}^n \) is an \( n \)-dim Cartesian or vector or linear space.

**Definition.** A region, or a region without boundary, ("open set") is a set \( D \) of points in \( \mathbb{R}^n \) such that together with each point \( P_0 \), \( D \) also contains all points sufficiently close to \( P_0 \), i.e.

\[
\forall P_0 = (x^1_0, \ldots, x^n_0) \in D \ \exists \epsilon > 0 : \\
\text{all points } P = (x^1, \ldots, x^n) \text{ satisfying } |x^i - x^i_0| < \epsilon, i = 1, \ldots, n \text{ also lie in } D.
\]

**Definition.** A *region with boundary* is obtained from a region \( D \) by adjoining all boundary points (i.e. points not in \( D \), yet having points of \( D \) arbitrarily close to them). The *boundary* of a region is just the set of boundary points.
$\mathbb{R}^n$ is an $n$-dim Cartesian or vector or linear space.

**Definition.** A region, or a region without boundary, ("open set") is a set $D$ of points in $\mathbb{R}^n$ such that together with each point $P_0$, $D$ also contains all points sufficiently close to $P_0$, i.e.

$$\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 :$$

all points $P = (x^1, \ldots, x^n)$ satisfying $|x^i - x_0^i| < \epsilon$, $i = 1, \ldots, n$ also lie in $D$.

**Definition.** A region with boundary is obtained from a region $D$ by adjoining all boundary points (i.e. points not in $D$, yet having points of $D$ arbitrarily close to them). The boundary of a region is just the set of boundary points.

**Definition.** $n$-dim *Euclidean* space is $\mathbb{R}^n$ with the distance $l$ between any two points given by

$$l^2 = \sum_{i=1}^{n} (x^i - y^i)^2$$
Definition 1.1.1. A differentiable $n$-dimensional manifold is a set $M$ (whose elements we call “points”) together with the following structure on it. The set $M$ is the union of a finite or countably infinite collection of subsets $U_q$ with the following properties

(i) Each subset $U_q$ has defined on it coordinates $x^\alpha_q$, $\alpha = 1, \ldots, n$ (called local coordinates) by virtue of which $U_q$ is identifiable with a region of Euclidean $n$-space $\mathbb{R}^n$ with Euclidean coordinates $x^\alpha_q$. The $U_q$ with their coordinate systems are called charts or local coordinate neighbourhoods.

(ii) Each non-empty intersection $U_p \cap U_q$ of a pair of charts thus has defined on it two coordinate systems, the restrictions of $(x^\alpha_p)$ and $(x^\alpha_q)$. It is required that under each of these coordinatisations the intersection $U_p \cap U_q$ is identifiable with a region of $\mathbb{R}^n$, and that each of these coordinate systems be expressible in terms of the other in a one-to-one differentiable manner. Thus, if the transition functions from $x^\alpha_q$ to $x^\alpha_p$ and back are given by

\begin{align}
    x^\alpha_p &= x^\alpha_p(x^1_q, \ldots, x^n_q), \quad \alpha = 1, \ldots, n, \\
    x^\alpha_q &= x^\alpha_q(x^1_p, \ldots, x^n_p), \quad \alpha = 1, \ldots, n, 
\end{align}

(1.1)

then in particular the Jacobian $\det(\partial x^\alpha_p/\partial x^\beta_q)$ is nonzero on $U_p \cap U_q$.

The general smoothness class of the transition functions for all intersecting pairs $U_p, U_q$ is called the smoothness class of the manifold $M$ with its accompanying atlas of charts $U_q$. 
Example 1. Any Euclidean space of regions is a manifold.

Example 2. A region of complex space $\mathbb{C}^n$ can be regarded as a region of $\mathbb{R}^{2n} \implies \mathbb{C}^n$ is a manifold.

Example 3. A 2-sphere $S^2$ is a manifold.

Example 4. A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.
Example 1. Any Euclidean space of regions is a manifold.

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Example 4. A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.

Example 5. Given two manifolds $M = \bigcup q U_q$, $N = \bigcup p V_p$ we construct their direct product $M \times N$ as follows:

The points of the manifold $M \times N$ are ordered pairs $(m, n)$, and covering by local coordinate neighbourhoods is given by

$$M \times N = \bigcup q,p U_q \times V_p$$

where if $x^\alpha_q$ are the coordinates on $U_q$ and $y^\beta_p$ on $V_p$ then the coordinates on $U_q \times V_p$ are $(x^\alpha_q, y^\beta_p)$.

E.g. $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times S^1$, $S^1 \times \mathbb{R}$, $S^1 \times S^1$, $\mathbb{R}^m \times \mathbb{R}^n$. 
2 Elements of Topology

The definition of manifold is very general. To restrict it we need some basic concepts of topology.

**Definition.** A topological space is a set \( X \) (of “points”) of which certain subsets, called the open sets of the topological space, are distinguished. These open sets have to satisfy:

(a) the intersection of any two (and hence of any finite collection) of them should again be an open set;

(b) the union of any collection of open sets must again be open;

(c) the empty set and the whole set \( X \) must be open.

The complement of any open set is called a closed set of the topological space.

In Euclidean space \( \mathbb{R}^n \) the “Euclidean topology” is the usual one where the open sets are the open regions.
**Definition.** A topological space is a set $X$ (of “points”) of which certain subsets, called the **open sets** of the topological space, are distinguished. These open sets have to satisfy:

(a) the intersection of any two (and hence of any finite collection) of them should again be an open set;

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The complement of any open set is called a **closed set** of the topological space.

In Euclidean space $\mathbb{R}^n$ the “Euclidean topology” is the usual one where the open sets are the open regions.

**Definition.** Given any subset $A \subseteq \mathbb{R}^n$, the **induced topology** on $A$ is that with open sets the intersections $A \cap U$ where $U$ ranges over all open sets of $\mathbb{R}^n$.

**Definition.** A map $f : X \mapsto Y$ of one topological space to another is **continuous** if the complete inverse image $f^{-1}(U)$ of every open set $U \subseteq Y$ is open in $X$.

**Definition.** Two topological spaces are **topologically equivalent** or **homeomorphic** if there is a one-to-one and onto map between them such that both it and its inverse are continuous.
**Definition 1.1.2.** The topology on a manifold $M$ is given by the following specification of the open sets.

In every local coordinate neighbourhood $U_q$ the open regions are to be open in the topology on $M$; the totality of open sets of $M$ is then obtained by admitting as open also arbitrary unions of countable collections of such regions, i.e. by closing under countable unions.

With this topology the continuous maps of a manifold $M$ turn out to be those which are continuous in the usual sense on each local coordinate neighbourhood $U_q$.

Any open subset $V$ of $M$ inherits, i.e. has induced on it, the structure of a manifold: $V = \bigcup_q V_q$ where $V_q = V \cap U_q$.  


**Definition.** A *metric space* is a set which comes equipped with a “distance function”, i.e. a real-valued function \( \rho(x, y) \) defined on pairs \( x, y \) of its elements (“points”), and having the following properties

(i) Symmetry: \( \rho(x, y) = \rho(y, x) \),

(ii) Positivity: \( \rho(x, x) = 0 \), \( \rho(y, x) > 0 \) if \( x \neq y \),

(iii) The triangle inequality: \( \rho(x, y) \leq \rho(x, z) + \rho(z, y) \),

**Example.** \( n \)-dim Euclidean space is a metric space with

\[
\rho(x, y) = \sqrt{\sum_{\alpha=1}^{n} (x^\alpha - y^\alpha)^2}.
\]

A metric space is topologised by taking as its open sets the unions of arbitrary collections of “open balls” where by *open ball* with centre \( x_0 \) and radius \( \epsilon \) we mean the set of all points \( x \) satisfying \( \rho(x, x_0) < \epsilon \).
Definition. A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centres at $x, y$ do not intersect.

All topological spaces we consider will be Hausdorff. In particular our manifolds will be Hausdorff spaces.
**Definition.** A topological space $X$ is said to be *compact* if every countable collection of open sets covering $X$ contains a finite subcollection already covering $X$.

If $X$ is a metric space then compactness is equivalent to the condition that from every sequence of points of $X$ a convergent subsequence can be selected.

**Definition.** A topological space is connected if any two points can be joined by a continuous path.
Definition 1.1.3. A manifold $M$ is said to be oriented if for every pair $U_p, U_q$ of intersecting local coordinate neighbourhoods the Jacobian $J_{pq} = \det(\partial x^\alpha_p / \partial x^\beta_q)$ of the transition functions is positive.

Euclidean $\mathbb{R}^n$ and $S^2$ are oriented.

Definition 1.1.4. We say that the coordinate systems $x$ and $y$ define the same orientation of $\mathbb{R}^n$ if $J > 0$ and opposite orientations if $J < 0$.

Euclidean $\mathbb{R}^n$ possesses two possible orientations, and any connected oriented manifold has exactly two orientations.
3 Mappings of Manifolds. Tensors on Manifolds

Let $M = \bigcup_p U_p$ with coordinates $x^\alpha_p$, and $N = \bigcup_q V_q$ with coordinates $y^\beta_q$ be two manifolds of dim $m$ and $n$.

**Definition 1.2.1.** A mapping $f : M \mapsto N$ is said to be smooth of smoothness class $k$ if for all $p, q$ for which $f$ determines functions $y_q^\beta(x_1^p, \ldots, x_m^p) = f(x_1^p, \ldots, x_m^p)_q^\beta$, these functions are, where defined, smooth of smoothness class $k$ (i.e. all their partial derivatives up to those of $k$th order exist and are continuous).

The smoothness class of $f$ cannot exceed the maximum class of the manifolds.

If $N = \mathbb{R}$ then $f$ is a real-valued function of the points of $M$.

A smooth mapping *may not* be defined on the whole manifold $M$.

E.g. each local coordinate $x^\alpha_p$ for fixed $p$ and $\alpha$ is a real-valued function on $M$ defined only on the region $U_p$.

**Definition 1.2.2.** The manifolds $M$ and $N$ are said to be *smoothly equivalent* or *diffeomorphic* if there is a one-to-one and onto map $f$ such that both $f : M \mapsto N$ and $f^{-1} : N \mapsto M$ are smooth of some class $k \geq 1$.

Since $f^{-1}$ exists then $J_{pq} = \det(\partial y^\beta_q / \partial x^\alpha_p) \neq 0$ wherever it is defined.

We always assume that the smoothness class of any manifolds and mappings are sufficiently high for our aims.
Let \( x = x(\tau) \), \( a \leq \tau \leq b \), be a curve segment on a manifold \( M \ni x(\tau) \).

In \( U_p \) with \( x^\alpha_p \) it is described by the parametric equations
\[
x^\alpha_p = x^\alpha_p(\tau), \quad \alpha = 1, \ldots, m,
\]
and in \( U_p \) its velocity vector (which is tangent to the curve) is
\[
\dot{x} = (\dot{x}^1_p, \ldots, \dot{x}^m_p).
\]

In \( U_p \cap U_q \) we have two representations \( x^\alpha_p(\tau) \) and \( x^\beta_q(\tau) \) of the curve and
\[
x^\alpha_p(x^1_q(\tau), \ldots, x^m_q(\tau)) = x^\alpha_p(\tau).
\]

Thus, velocities in the two systems are related as
\[
\dot{x}^\alpha_p = \sum_{\beta=1}^{m} \frac{\partial x^\alpha_p}{\partial x^\beta_q} \dot{x}^\beta_q \equiv \frac{\partial x^\alpha_p}{\partial x^\beta_q} \dot{x}^\beta_q \quad \forall \alpha,
\]
where we use Einstein’s summation rule:
- each index can appear at most \textit{twice} in any term;
- \textit{repeated} indices are implicitly summed over;
- one repeated index must be \textit{upper} and the other one must be \textit{lower}. 

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**Definition 1.2.3.** A *tangent* vector to an $m$-dim manifold $M$ at an arbitrary point $x$ is represented in terms of local coordinates $x^\alpha_p$ by an $m$-tuple $(\xi^\alpha)$ of components which are linked to the components in terms of any other system $x^\beta_q$ of local coordinates as

$$
\xi^\alpha_p = \left( \frac{\partial x^\alpha_p}{\partial x^\beta_q} \right)_x \xi^\beta_q \quad \forall \, \alpha, \tag{3.2}
$$

The set of all tangent vectors to an $m$-dim manifold $M$ at a point $x$ forms an $m$-dim vector space $T_x = T_x M$, the *tangent space* to $M$ at the point $x$.

Thus, the velocity vector at $x$ of any smooth curve on $M$ through $x$ is a tangent vector to $M$ at $x$.

From (3.2) one sees that for any choice of local coordinates $x^\alpha$ in a neighbourhood of $x$, the operators $\frac{\partial}{\partial x^\alpha}$, operating on real-valued functions on $M$, may be thought of as forming a basis $e_\alpha = \frac{\partial}{\partial x^\alpha}$ for the tangent space $T_x$

$$(3.2) \quad \implies \quad \xi^\alpha_p \frac{\partial}{\partial x^\alpha_p} = \xi^\beta_q \frac{\partial}{\partial x^\beta_q}.$$
**Definition.** A smooth map $f$ from $M$ to $N$ gives rise for each $x$ to a push-forward or an induced linear map of tangent spaces

$$f_* : T_x M \mapsto T_{f(x)} N,$$

defined as sending the velocity vector at $x$ of any smooth curve $x = x(\tau)$ on $M$ to the velocity vector at $f(x)$ to the curve $f(x(\tau))$ on $N$.

In terms of local coordinates $x^\alpha$ in a neighbourhood of $x \in M$, and $y^\beta$ in a neighbourhood of $f(x) \in N$ the map $f$ is written as

$$y^\beta = f^\beta(x^1, \ldots, x^m), \quad \beta = 1, \ldots, n,$$

and the push-forward map $f_*$ as

$$\xi^\alpha \mapsto \eta^\beta = \frac{\partial f^\beta}{\partial x^\alpha} \xi^\alpha.$$

For a real-valued function $f : M \mapsto \mathbb{R}$, the push-forward map $f_*$ corresponding to each $x \in M$ is a real-valued linear function on the tangent space to $M$ at $x$

$$\xi^\alpha \mapsto \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha,$$

and it is represented by the gradient of $f$ at $x$, and is a co-vector or one-form. Thus, $f_*$ can be identified with the differential $df$. In particular

$$dx^\alpha_p : \xi^\alpha \mapsto \eta = \xi^\alpha_p.$$
Definition 1.2.4. A **Riemann metric** on a manifold $M$ is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local coordinates of the points. Thus, at each point $x = (x_1^p, \ldots, x_m^p)$ of each chart $U_p$, the metric is given by a symmetric matrix $(g_{\alpha\beta}^{(p)}(x_1^p, \ldots, x_m^p))$, and determines a symmetric scalar product of pairs of tangent vectors at the point $x$

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)}(x_1^p, \ldots, x_m^p) \xi_{\alpha}^p \eta_{\beta}^p = \langle \eta, \xi \rangle, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be coordinate-independent

$$g_{\alpha\beta}^{(p)} \xi_{\alpha}^p \eta_{\beta}^p = g_{\alpha\beta}^{(q)} \xi_{\alpha}^q \eta_{\beta}^q$$

and therefore the coefficients $g_{\alpha\beta}^{(p)}$ of the quadratic form transform as

$$g_{\gamma\delta}^{(q)} = \frac{\partial x_\alpha^p}{\partial x_\gamma^q} \frac{\partial x_\beta^p}{\partial x_\delta^q} g_{\alpha\beta}^{(p)}$$

For a **pseudo-Riemann** metric on $M$ one just requires the quadratic form to be nondegenerate.
Definition 1.2.4. A *tensor of type* \((k, l)\) and *rank* \(k + l\) on an \(m\)-dim manifold \(M\) is given in each local coordinate system \(x^\alpha_p\) by a family of functions

\[
^{(p)}T^{i_1\ldots i_k}_{j_1\ldots j_l}(x)
\]

of the point \(x\).

In other local coordinates \(x^\beta_q\) the components \(^{(q)}T^{s_1\ldots s_k}_{t_1\ldots t_l}(x)\) of the same tensor are

\[
^{(q)}T^{s_1\ldots s_k}_{t_1\ldots t_l} = \frac{\partial x^s_1}{\partial x^i_p} \cdot \frac{\partial x^s_k}{\partial x^i_p} \cdot \frac{\partial x^j_1}{\partial x^t_q} \cdot \frac{\partial x^j_l}{\partial x^t_q} \cdot ^{(p)}T^{i_1\ldots i_k}_{j_1\ldots j_l} \tag{3.3}
\]

Let’s rewrite (3.3) as

\[
^{(q)}T^{s_1\ldots s_k}_{t_1\ldots t_l} dx^t_1 \cdots dx^t_l \frac{\partial}{\partial x^s_1} \cdots \frac{\partial}{\partial x^s_k} = ^{(p)}T^{i_1\ldots i_k}_{j_1\ldots j_l} dx^j_1 \cdots dx^j_l \frac{\partial}{\partial x^i_p} \cdots \frac{\partial}{\partial x^i_p}
\]
4 Algebraic Operations on Tensors (vol 1, section 17)

1. **Permutation of indices.** Let \( \sigma \) be some permutation of \( 1, 2, \ldots, l \)

\[
\sigma = \begin{pmatrix}
1 & \ldots & l \\
\sigma(1) & \ldots & \sigma(l)
\end{pmatrix}
\]

\( \sigma \) acts on the ordered \( l \)-tuple \((j_1, \ldots, j_l)\) as

\[
\sigma(j_1, \ldots, j_l) = (j_{\sigma 1}, \ldots, j_{\sigma l})
\]

We say that a tensor \( \tilde{T}_{i_1 \ldots i_k} \) is obtained from a tensor \( T_{j_1 \ldots j_l} \) by means of a permutation \( \sigma \) of the lower indices if at each point of \( M \)

\[
\tilde{T}_{i_1 \ldots i_k} = T_{\sigma(j_1 \ldots j_l)}
\]

Permutations of the upper indices are defined similarly.

**Example.** \( \tilde{T}_{ij} = T_{\sigma(ij)} = T_{ji} \) which is a matrix transposition.
1. **Permutation of indices.** Let $\sigma$ be some permutation of $1, 2, \ldots, l$

$$\sigma = \begin{pmatrix} 1 & \ldots & l \\ \sigma(1) & \ldots & \sigma(l) \end{pmatrix}$$

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

$$\sigma(j_1, \ldots, j_l) = (j_{\sigma_1}, \ldots, j_{\sigma_l})$$

We say that a tensor $\tilde{T}^{\cdot 1 \cdots \cdot k}_{\cdot 1 \cdots \cdot l}$ is obtained from a tensor $T^{\cdot 1 \cdots \cdot k}_{\cdot 1 \cdots \cdot l}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$\tilde{T}^{\cdot 1 \cdots \cdot k}_{\cdot 1 \cdots \cdot l} = T^{\cdot 1 \cdots \cdot k}_{\cdot 1 \cdots \cdot l}$$

Permutations of the upper indices are defined similarly.

**Example.** $\tilde{T}_{ij} = T_{\sigma(ij)} = T_{ji}$ which is a matrix transposition.

2. **Contraction (taking “traces”).** By the contraction of a tensor $T^{\cdot 1 \cdots \cdot k}_{\cdot 1 \cdots \cdot l}$ of type $(k, l)$ with respect to the indices $i_a, j_b$ we mean the tensor (summation over $n$)

$$\tilde{T}^{\cdot 1 \cdots \cdot k-1}_{\cdot 1 \cdots \cdot l-1} = T^{\cdot 1 \cdots \cdot i_{a-1} i_{a+1} \cdots \cdot k}_{\cdot 1 \cdots \cdot j_{b-1} j_{b+1} \cdots \cdot l}$$

of type $(k - 1, l - 1)$.

**Example.** $T^n = \text{tr} T$ of the matrix $T^i_j$.  

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1. Permutation of indices. Let $\sigma$ be some permutation of $1, 2, \ldots, l$

$$\sigma = \begin{pmatrix} 1 & \ldots & l \\ \sigma(1) & \ldots & \sigma(l) \end{pmatrix}$$

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

$$\sigma(j_1, \ldots, j_l) = (j_{\sigma(1)}, \ldots, j_{\sigma(l)})$$

We say that a tensor $\widetilde{T}_{i_1 \cdots i_k}$ is obtained from a tensor $T_{i_1 \cdots i_k}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$\widetilde{T}_{i_1 \cdots i_k} = T_{\sigma(j_1 \cdots j_l)}$$

Permutations of the upper indices are defined similarly.

**Example.** $\widetilde{T}_{ij} = T_{\sigma(ij)} = T_{ji}$ which is a matrix transposition.

2. Contraction (taking “traces”). By the contraction of a tensor $T_{i_1 \cdots i_k}$ of type $(k, l)$ with respect to the indices $i_a, j_b$ we mean the tensor (summation over $n$)

$$\widetilde{T}_{i_1 \cdots i_{k-1}} = T_{i_1 \cdots i_{a-1} n i_a + 1 \cdots i_k}$$

of type $(k-1, l-1)$.

**Example.** $T^n_n = \text{tr} T$ of the matrix $T^i_j$.

3. Product of tensors. Given two tensors $T = (T_{i_1 \cdots i_p}^{j_1 \cdots j_q})$ of type $(p, q)$ and $P = (P_{i_1 \cdots i_k}^{j_1 \cdots j_l})$ of type $(k, l)$, we define their product to be the tensor $S = T \otimes P$ of type $(p+k, q+l)$ with components

$$S_{i_1 \cdots i_{p+k}}^{j_1 \cdots j_{q+l}} = T_{i_1 \cdots i_p}^{j_1 \cdots j_q} P_{j_{p+1} \cdots j_{p+k}}^{i_{q+1} \cdots i_{q+l}}$$

This multiplication is **not** commutative but it is associative.
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^i_j = \xi^i \eta_j$ of type $(1, 1)$. Contraction $T^i_i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_l \Rightarrow T^{ik}_l = \xi^i A^k_l$ of type $(2, 1)$. Contraction $\eta^k = \xi^i A^k_i$ is a vector, the result of applying the linear transformation $A^k_l$ to the vector.
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^i_j = \xi^i \eta_j$ of type $(1, 1)$. Contraction $T^i_i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_i \Rightarrow T^i_{ik} = \xi^i A^k_i$ of type $(2, 1)$. Contraction $\eta^k = \xi^i A^k_i$ is a vector, the result of applying the linear transformation $A^k_i$ to the vector.

Example 3. We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows:
Since the gradient $\frac{\partial f}{\partial x^i}$ of a function $f$ is a co-vector, the quantity
$$\partial_\xi f = \xi^i \frac{\partial f}{\partial x^i}$$
is a scalar called the directional derivative of $f$ in the direction of $\xi$.
Thus, an arbitrary vector $\xi$ corresponds to the operator
$$\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$$
We identify $e_i = \frac{\partial}{\partial x^i}$ with the canonical basis of the tangent space.
5 Tensors of type \( (0, k) \) (vol 1, section 18)

These are tensors with lower indices: \( T_{i_1 \cdots i_k} \)

5.1 Co-vectors: Tensors of type \( (0, 1) \)

The gradient \( \left( \frac{\partial f}{\partial x^i} \right) \) of a function \( f \) is the standard example.

Recall that the differential of a function \( f \) of \( x^1, \ldots, x^n \) corresponding to increments \( dx^i \) in the \( x^i \) is

\[
df = \frac{\partial f}{\partial x^i} dx^i
\]

Since \( dx^i \) is a vector \( df \) has the same value in any coordinate system.

In general, given any co-vector \( (T_i) \), the differential form \( T_i dx^i \) is invariant under a change of a chart.

We identify \( dx^i \equiv e^i \) with the canonical basis of co-vectors or cotangent space.
5.2 Tensors of type $(0, 2)$

A basis for the space of tensors of type $(0, 2)$ at a given point are the products

$$e^i \otimes e^j$$

In terms of this basis an arbitrary tensor $T_{ij}$ has the form

$$T_{ij} e^i \otimes e^j$$

and can be regarded as a bilinear form on vectors since if $\xi$, $\eta$ are vectors then the scalar

$$T_{ij} \xi^i \eta^j$$

can be considered as the value of the bilinear form on those vectors.
Any $T_{ij}$ can be expressed as

$$T_{ij} = T_{ij}^{\text{sym}} + T_{ij}^{\text{alt}}$$

$$T_{ij}^{\text{sym}} = \frac{1}{2}(T_{ij} + T_{ji}) = T_{ji}^{\text{sym}}$$

$$T_{ij}^{\text{alt}} = \frac{1}{2}(T_{ij} - T_{ji}) = -T_{ji}^{\text{alt}}$$

A basis of $T_{ij}^{\text{sym}}$ is $\frac{e^i \otimes e^j + e^j \otimes e^i}{2}, \quad i \leq j$

A basis of $T_{ij}^{\text{alt}}$ is $e^i \otimes e^j - e^j \otimes e^i, \quad i < j$

Then

$$T_{ij}^{\text{sym}} e^i \otimes e^j = T_{ij}^{\text{sym}} \frac{e^i \otimes e^j + e^j \otimes e^i}{2}$$

$$= \sum_i T_{ii}^{\text{sym}} e^i \otimes e^i + \sum_{i<j} 2T_{ij}^{\text{sym}} \frac{e^i \otimes e^j + e^j \otimes e^i}{2}$$

and

$$T_{ij}^{\text{alt}} e^i \otimes e^j = T_{ij}^{\text{alt}} \frac{e^i \otimes e^j - e^j \otimes e^i}{2}$$

$$= \sum_{i<j} T_{ij}^{\text{alt}} (e^i \otimes e^j - e^j \otimes e^i)$$

In differential notation we identify

$$\frac{e^i \otimes e^j + e^j \otimes e^i}{2} \equiv dx^i dx^j = dx^j dx^i$$

$$e^i \otimes e^j - e^j \otimes e^i \equiv dx^i \wedge dx^j = -dx^j \wedge dx^i$$
5.3 Skew-symmetric Tensors of type $(0, k)$

**Definition.** A *skew-symmetric* tensor of type $(0, k)$ is a tensor $T_{i_1 \cdots i_k}$ satisfying

$$T_{\sigma(i_1 \cdots i_k)} = s(\sigma)T_{i_1 \cdots i_k}$$

where for all permutations $\sigma$

$$s(\sigma) = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \end{cases}$$

Thus, if two indices are equal then the corresponding component of $T_{i_1 \cdots i_k}$ is equal to 0.

Then, if $k > n$ the tensor is identically 0.

In what follows we assume $k \leq n$. 
The standard basis at a given point is
\[ dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k \]
where
\[ dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathbf{s}(\sigma) e^{\sigma(i_1)} \otimes \cdots \otimes e^{\sigma(i_k)} \]
Here \( S_k \) is the symmetric group,
i.e. the group of all permutations of \( 1, \ldots, k \),
and \( \sigma(i_l) \equiv i_{\sigma(l)} \).
The differential form of the skew-symmetric tensor \( (T_{i_1 \cdots i_k}) \) is
\[ T_{i_1 \cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = \sum_{i_1 < \cdots < i_k} T_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]
\[ = \frac{1}{k!} T_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]
**Example 1.** A skew-symmetric tensor $T_{i_1\ldots i_n}$ of type $(0, n)$ in $n$-dim manifold is determined by the single component $T_{1\ldots n}$

$$T_{\sigma(1\ldots n)} = s(\sigma)T_{1\ldots n}$$

Thus, the space of skew-symmetric tensors of type $(0, n)$ is one-dim.

We denote $T_{i_1\ldots i_n}$ with $T_{1\ldots n} = 1$ as

$$\epsilon_{i_1\ldots i_n}$$

It is called the **Levi-Civita** symbol (or tensor) of rank $n$.

**Theorem.** Skew-symmetric tensors of type $(0, n)$ where $n$ is the dimension of the manifold $M$ transform as

$$(p)T_{1\ldots n} = (q)T_{1\ldots n} \cdot J$$

where $J$ is the Jacobian

$$J = \det \left( \frac{\partial x^i_q}{\partial x^j_p} \right)$$

HW: Prove the theorem
Example 2.
Let \( G = (g_{ij}) \) be a non-degenerate tensor, i.e. \( g \equiv \det(g_{ij}) \neq 0 \) (it does not have to be symmetric).

Then
\[
g_{ij}^{(p)} = \frac{\partial x^k_q}{\partial x^l_p} \frac{\partial x^l_q}{\partial x^j_p} g_{kl}^{(q)} = \frac{\partial x^k_q}{\partial x^l_p} g_{kl}^{(q)} \frac{\partial x^l_q}{\partial x^j_p}
\]
or in matrix notations
\[
g_{ij}^{(p)} = (A^T G^{(q)} A)_{ij}, \quad A = (A^l_j) = \left( \frac{\partial x^l_q}{\partial x^j_p} \right) \implies g^{(p)} = (\det A)^2 g^{(q)}
\]

Thus, if \( \det A > 0 \) then
\[
\sqrt{|g^{(p)}|} = \sqrt{|g^{(q)}|} \det A = \sqrt{|g^{(q)}|} J
\]

Comparing with
\[
^{(p)} T_{1...n} = ^{(q)} T_{1...n} \cdot J
\]

one concludes

The expression \( \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^n \) behaves as a tensor under coordinate changes for which the Jacobian \( J = \det \left( \frac{\partial x^i_q}{\partial x^j_p} \right) \) is positive.

Since \( J > 0 \), the manifold \( M \) is oriented.
Metric and distance function.

A metric $g_{ij}$ on a manifold is a tensor of type $(0, 2)$, and on an oriented manifold such a metric gives rise to a volume element

$$T_{i_1\ldots i_k} = \sqrt{|g|} \epsilon_{i_1\ldots i_k}, \quad g = \det(g_{ij})$$

It is convenient to write the volume element in the notation of differential forms

$$\Omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

If $g_{ij}$ is Riemann then the volume $V$ of $M$ is

$$V = \int_M \Omega = \int_M \sqrt{g} dx^1 \wedge \cdots \wedge dx^n$$

A Riemann metric $ds^2 = g_{ij}dx^i dx^j$ on a connected manifold $M$ gives rise to a metric space structure on $M$ with distance function $\rho(x, y)$ defined by

$$\rho(x, y) = \inf_{\gamma} \int_{\gamma} ds$$

where the infimum is taken over all piece-wise smooth arcs joining the points $x$ and $y$.

The topology on $M$ defined by this metric-space structure coincides with the Euclidean topology on $M$.

6 The behaviour of Tensors under Mappings (vol 1, section 22)

6.1 Tensors of type $(k, 0)$

Recall that a smooth map $f$ from an $m$-dim manifold $M$ to an $n$-dim manifold $N$ gives rise $\forall x \in M$ to the push-forward map of tangent spaces

$$f_* : T_x M \mapsto T_{f(x)} N$$
which in terms of coordinates $x^i$ in $U \subset M$, $x \in U$, and $y^a$ in $V \subset N$, $y \in U$ is written as

$$y^a = f^a(x^1, \ldots, x^m), \quad a = 1, \ldots, n$$

$$f_* : \xi^i \mapsto \eta^a = \frac{\partial f^a}{\partial x^i} \xi^i$$

This can be generalised to a push-forward map of the spaces of tensors of type $(k, 0)$

$$f_* : \xi^{i_1 \cdots i_k} \mapsto \eta^{a_1 \cdots a_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi^{i_1 \cdots i_k}$$
6.2 Tensors of type \((0, k)\)

Let \(T^{(0,k)}_x M\) denote the space of tensors of type \((0, k)\) at \(x \in M\). Let \(f\) be a smooth map \(f\) from an \(m\)-dim manifold \(M\) to an \(n\)-dim manifold \(N\). It gives rise to a map

\[ f^* : T^{(0,k)}_{f(x)} N \rightarrow T^{(0,k)}_x M \]

which in terms of coordinates \(x^i\) in \(U \subset M\), \(x \in U\), and \(y^a\) in \(V \subset N\), \(y \in U\) is written as

\[ f^* : \eta_{a_1 \cdots a_k} \mapsto \xi_{i_1 \cdots i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1 \cdots a_k} \]

The map \(f^*\) is called the pullback.

Let us denote

\[ \zeta(\theta) \equiv \zeta_{i_1 \cdots i_k} \theta^{i_1 \cdots i_k} \]

the full contraction of a tensor of type \((0, k)\) with a tensor of type \((k, 0)\).

If \(\xi_{i_1 \cdots i_k}\) and \(\tilde{\eta}_{a_1 \cdots a_k}\) are tensors of type \((k, 0)\) and \((0, k)\) in \(T^{(k,0)}_x M\) and \(T^{(0,k)}_y N\), respectively, then

\[
(f^* \tilde{\eta})(\xi) = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \tilde{\eta}_{a_1 \cdots a_k} \xi_{i_1 \cdots i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi_{i_1 \cdots i_k} = \tilde{\eta}(f^* \xi)
\]
Embeddings and Immersions of Manifolds

**Definition 1.3.1a.** A manifold $M$ of dim $m$ is said to be *immersed* in a manifold $N$ of dim $n \geq m$ if $\exists$ a smooth map $f : M \hookrightarrow N$ such that the push-forward map $f_*$ is at each point a one-to-one map of the tangent space. The map $f$ is called an *immersion* of $M$ into $N$.

Since $f_*$ is at each point a one-to-one map of the tangent space, in terms of local coordinates the Jacobian matrix of $f$ at each point has rank equal to $m = \dim M$.

**Definition 1.3.1b.** An immersion of $M$ into $N$ is called *embedding* if it is one-to-one. Then, $M$ is called a *submanifold* of $N$. 
Example 1. Suppose $M$ is an $m$-dim submanifold of an $n$-dim (pseudo-)Riemann manifold $N$ with metric $g^{(N)}_{ab}$. Then, the pullback of $g^{(N)}_{ab}$ to $M$ yields the tensor

$$g^{(M)}_{ij}(x) = \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} g^{(N)}_{ab}(f(x)), \quad i, j = 1, \ldots, m, \quad a, b = 1, \ldots, n$$

which is called the metric induced on the submanifold $M$ by the metric $g^{(N)}_{ab}$ of $N$.

Consider the line element $ds_N$ of $N$

$$ds_N^2 = g^{(N)}_{ab}(y) dy^a dy^b$$

Let

$$y^a = f^a(x^1, \ldots, x^m), \quad a = 1, \ldots, n$$

Then, the line element becomes

$$ds_N^2 = g^{(N)}_{ab}(f(x)) \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} dx^i dx^j = g^{(M)}_{ij}(x) dx^i dx^j = ds_M^2$$

Thus, the infinitesimal distances measured by using the metric on $N$ and the metric induced on $M$ are the same.

A simple example is $S^2$ in $\mathbb{R}^3$. 

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Example 2. Consider the pullback of a skew-symmetric tensor $T_{a_1 \cdots a_m}$ of type $(0, m)$, i.e. an $m$-form, to an $m$-dim submanifold $M$, $y^a = f^a(x^1, \ldots, x^m)$, $a = 1, \ldots, n$ in an $n$-dim manifold $N$.

Theorem 22.1.2. The pullback of the skew-symmetric form

$$\frac{1}{m!} T_{a_1 \cdots a_m} dy^{a_1} \wedge \cdots \wedge dy^{a_m}$$

to the $m$-dim submanifold $y^a = f^a(x^1, \ldots, x^m)$ is given by

$$\left( \frac{1}{m!} J^{a_1 \cdots a_m} T_{a_1 \cdots a_m} \right) dx^1 \wedge \cdots \wedge dx^m$$

where $J^{a_1 \cdots a_m}$ is the $m \times m$ minor of the matrix $(\partial y^a / \partial x^i)$ formed from the columns numbered $a_1, \ldots, a_m$.

Thus, we have on $M$

$$\frac{1}{m!} T_{a_1 \cdots a_m} dy^{a_1} \wedge \cdots \wedge dy^{a_m} = \left( \frac{1}{m!} J^{a_1 \cdots a_m} T_{a_1 \cdots a_m} \right) dx^1 \wedge \cdots \wedge dx^m$$

Proof. By definition of the pullback

$$\tilde{T}_{a_1 \cdots a_m} = \frac{\partial y^{a_1}}{\partial x^1} \cdots \frac{\partial y^{a_m}}{\partial x^m} T_{a_1 \cdots a_m}$$

$$= \sum_{a_1 < \cdots < a_m} T_{a_1 \cdots a_m} \left( \sum_{\sigma \in S_m} \varepsilon(\sigma) \frac{\partial y^{a_{\sigma(1)}}}{\partial x^1} \cdots \frac{\partial y^{a_{\sigma(m)}}}{\partial x^m} \right)$$

$$= \sum_{a_1 < \cdots < a_m} T_{a_1 \cdots a_m} J^{a_1 \cdots a_m} = \frac{1}{m!} J^{a_1 \cdots a_m} T_{a_1 \cdots a_m}$$
7 Embeddings and Immersions of Manifolds

Definition 1.3.1a. A manifold $\mathcal{M}$ of dim $m$ is said to be immersed in a manifold $\mathcal{N}$ of dim $n \geq m$ if $\exists$ a smooth map $f : \mathcal{M} \rightarrow \mathcal{N}$ such that the push-forward map $f_*$ is at each point a one-to-one map of the tangent space. The map $f$ is called an immersion of $\mathcal{M}$ into $\mathcal{N}$.

Since $f_*$ is at each point a one-to-one map of the tangent space, in terms of local coordinates the Jacobian matrix of $f$ at each point has rank equal to $m = \dim \mathcal{M}$.

Definition 1.3.1b. An immersion of $\mathcal{M}$ into $\mathcal{N}$ is called embedding if it is one-to-one. Then, $\mathcal{M}$ is called a submanifold of $\mathcal{N}$. 
We always assume that any submanifold $M$ is defined in each chart $U_p$ of the containing manifold $N$ by a system of eqs
\[
\begin{align*}
&f^1_p(x^1_p, \ldots, x^n_p) = 0 \\
&f^2_p(x^1_p, \ldots, x^n_p) = 0 \\
&\vdots \\
&f^{n-m}_p(x^1_p, \ldots, x^n_p) = 0
\end{align*}
\]
where $\text{rank} \left( \frac{\partial f^i_p}{\partial x^\alpha_p} \right) = n - m$

with the property that on each intersection $U_p \cap U_q$ the systems $(f^i_p = 0)$ and $(f^i_q = 0)$ have the same set of zeroes.

Let us introduce in each $U_p \subset N$ new coordinates $y^1_p, \ldots, y^n_p$ satisfying
\[
y^m_{p+1} = f^1_p(x^1_p, \ldots, x^n_p), \quad y^{m+2}_p = f^2_p(x^1_p, \ldots, x^n_p), \ldots, \quad y^n_p = f^{n-m}_p(x^1_p, \ldots, x^n_p)
\]

Then, $M$ is given by
\[
y^m_{p+1} = 0, \quad y^{m+2}_p = 0, \quad \ldots, \quad y^n_p = 0
\]
while $y^1_p, \ldots, y^m_p$ serve as local coordinates on $M$. 


Definition 1.3.2. A closed region $A$ of a manifold $M$ defined by an inequality $f(x) \leq 0$ (or $f(x) \geq 0$) where $f$ is a real-valued function on $M$ is called a *manifold with boundary*.

It is assumed that the boundary $\partial A$ given by $f(x) = 0$ is a non-singular submanifold of $M$, i.e. $\vec{\nabla} f \neq 0$ on $\partial A$.

Let $A$ and $B$ be manifolds with boundary, both given as closed regions of manifolds $M$ and $N$. A map

$$\varphi : A \mapsto B$$

is said to be a *smooth map of manifolds with boundary* if it is a restriction to $A$ of a smooth map

$$\tilde{\varphi} : U \mapsto N, \quad \tilde{\varphi}|A = \varphi$$

of an open region $U$ of $M$ containing $A$, e.g. if

$$A : f(x) \leq 0 \text{ then } U \text{ is } U_\epsilon = \{x|f(x) < \epsilon\}, \quad \epsilon > 0.$$ 

**Definition.** A compact manifold without boundary is called *closed.*
8 Surfaces in Euclidean space

8.1 Surfaces as Manifolds

**Definition.** A non-singular surface $M$ of dimension $k$ in $n$-dim Euclidean space is given by a set of $n - k$ eqs

$$f_i(x^1, \ldots, x^n) = 0, \quad i = 1, \ldots, n - k$$  \hspace{1cm} (8.4)

where \forall x the matrix $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ has rank $n - k$.

Let $J_{j_1 \ldots j_{n-k}}$ be the minor of a submatrix made up of the columns of $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ which are indexed by $j_1, \ldots, j_{n-k}$.

Let $U_{j_1 \ldots j_{n-k}}$ be the region consisting of all points of the surface at which $J_{j_1 \ldots j_{n-k}}$ does not vanish.

Obviously,

$$M = \bigcup_{j_1 \ldots j_{n-k}} U_{j_1 \ldots j_{n-k}}$$

Since $J_{j_1 \ldots j_{n-k}} \neq 0$ on $U_{j_1 \ldots j_{n-k}}$, we can take

$$(y^1, \ldots, y^k) = (x^1, \ldots, \hat{x}^{j_1}, \ldots, \hat{x}^{j_{n-k}}, \ldots, x^n)$$  \hspace{1cm} (8.5)

as local coordinates on $U_{j_1 \ldots j_{n-k}}$.

**Theorem 2.1.1.** The covering of the surface $M$ (8.4) by the regions $U_{j_1 \ldots j_{n-k}}$, \hspace{1cm} $1 \leq j_1 < \cdots < j_{n-k} \leq n$

each furnished with local coordinates (8.5), defines on the surface the structure of a smooth manifold.

**Proof.** Blackboard
**Remark 1.** The Jacobian of the transition function \( y \to z \) is

\[
J_{(y)\to(z)} = \pm \frac{J_{s_1 \cdots s_{n-k}}}{J_{j_1 \cdots j_{n-k}}}
\]

HW: Prove it.

**Remark 2.** The tangent space to the surface \( M \) (8.4) is identifiable with the linear subspace of \( \mathbb{R}^n \) consisting of the solutions of the system

\[
\frac{\partial f_1}{\partial x^\alpha} \xi^\alpha = 0, \ldots, \frac{\partial f_{n-k}}{\partial x^\alpha} \xi^\alpha = 0
\]

Thus, the co-vectors \( \vec{\nabla} f_i = \left( \frac{\partial f_i}{\partial x^\alpha} \right), i = 1, \ldots, n - k \) are orthogonal to the surface at each point.
8.2 Surfaces can be oriented

Consider at any point $x$ of an $n$-dim manifold $M$ the various frames (i.e. ordered bases)

$$\tau = (e_1, \ldots, e_n)$$

for the tangent space to $M$ at $x$. Any two such frames

$$\tau_1 = (e^{(1)}_1, \ldots, e^{(1)}_n) \quad \text{and} \quad \tau_2 = (e^{(2)}_1, \ldots, e^{(2)}_n)$$

are related by a nonsingular linear transformation $A$

$$A : e^{(1)}_k \rightarrow e^{(2)}_k, \quad k = 1, \ldots, n$$

**Definition.** We say the ordered bases $\tau_1$, $\tau_2$ lie in the same orientation class if $\det A > 0$, and lie in opposite orientation classes if $\det A < 0$.

**Definition 2.1.2.** A manifold is said to be orientable if it is possible to choose at every point of it a single orientation class depending continuously on the points.

A particular choice of such an orientation class for each point is called an orientation of the manifold, and a manifold equipped with a particular orientation is said to be oriented.

If no orientation exists the manifold is non-orientable.
**Theorem 2.1.3.** Definition 1.1.3 is equivalent to Definition 2.1.2.

**Proof of Def 1.1.3 ⇒ Def 2.1.2.**
Let $M$ be oriented in the sense of Def 1.1.3. Then, we choose at each

$$x \in U_j \subset M$$

as our orienting frame the $n$-tuple

$$(e_{1j}, \ldots, e_{nj})$$

consisting of the standard basis vectors tangent to the coordinate axes of the local coordinate system

$$x_j^1, \ldots x_j^n$$

If

$$x \in U_j \quad \text{and} \quad x \in U_k$$

then the two orienting frames are related by the Jacobian matrix of transition function. Since the Jacobian is positive the two frames lie in the same orientation class.

**Proof of Def 1.1.3 ⇒ Def 2.1.2.** See the textbook
**Theorem 2.1.4.** A smooth non-singular surface $M^k$ in $n$-dim space $\mathbb{R}^n$, defined by a system of eqs (8.4), is orientable.

**Proof.** Let $\tau$ denote a point-dependent tangent frame to the surface $M^k$. The ordered $n$-tuple

$$\hat{\tau} = (\tau, \nabla f_1, \ldots, \nabla f_{n-k})$$

of vectors is linearly independent at each point because $\nabla f_i$ are linearly independent among themselves and orthogonal to the surface.

We can choose $\tau$ at each $x \in M^k$ so that $\hat{\tau}$ lies in the same orientation class as the standard frame

$$(e_1, \ldots, e_n)$$

Since this orientation class depends continuously on $x \in \mathbb{R}^n$, so will the orientation class of $\tau$ depend continuously on $x \in M^k$. 

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Example 1. An $n$-sphere $S^n$ in $\mathbb{R}^{n+1}$: $x_1^2 + \cdots + x_{n+1}^2 = 1$.

The $n$-sphere bounds a manifold with boundary, denoted by $D^{n+1}$ and called the closed $(n + 1)$-dim disc or ball, defined by

$$f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1 \leq 0$$

Then $S^n$ separates $\mathbb{R}^{n+1}$ into two non-intersecting regions defined by $f(x) < 0$ and $f(x) > 0$.

Example 2. A hyperbolic $n$-space, $H^n$, is one sheet of the hyperboloid of two sheets realised as a surface

$$-x_0^2 + \sum_{i=1}^n x_i^2 = -1, \quad x_0 > 0,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 3. An $n$-dimensional de Sitter space, $dS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_0^2 + \sum_{i=1}^n x_i^2 = 1,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 4. An $n$-dimensional anti-de Sitter space, $AdS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_0^2 - x_{-1}^2 + \sum_{i=1}^{n-1} x_i^2 = -1,$$

in the pseudo-Euclidean space $\mathbb{R}^{2,n-1}$
Definition 2.1.5. A connected \((n - 1)\)-dim submanifold of \(\mathbb{R}^n\) is called two-sided if a single-valued continuous field of unit normals can be defined on it. Such a submanifold is called a two-sided hypersurface.

Theorem 2.1.6. A two-sided hypersurface in \(\mathbb{R}^n\) is orientable.

Proof. See the textbook

It can be shown that any two-sided hypersurface in \(\mathbb{R}^n\) is defined by a single non-singular eq \(f(x) = 0\). Thus, it bounds a manifold with boundary.

Then, one can prove that any closed hypersurface in \(\mathbb{R}^n\) is two-sided.
8.3 Transformation Groups as Surfaces

**Definition.** A *group* is a nonempty set $G$ on which there is defined a binary operation $(a, b) \mapsto ab$ satisfying the following properties

- **Closure:** If $a$ and $b$ belong to $G$, then $ab$ is also in $G$.
- **Associativity:** $a(bc) = (ab)c$ for all $a, b, c \in G$.
- **Identity:** There is an element $1 \in G$ such that $a1 = 1a = a$ for all $a$ in $G$.
- **Inverse:** If $a \in G$, then there is an element $a^{-1} \in G$:
  $$aa^{-1} = a^{-1}a = 1.$$

Examples of groups which are manifolds are

1. The *general* linear group $GL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with non-zero determinant is a region in $\mathbb{R}^{n^2}$

2. The *special* linear group $SL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with determinant equal to 1
  $$\det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$
  is a hypersurface in $\mathbb{R}^{n^2}$

3. The *orthogonal* group $O(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying
  $$A^T \cdot A = I, \quad A \in \text{Mat}(n, \mathbb{R})$$
  is a surface in $\mathbb{R}^{n^2}$

4. The *special orthogonal* group $SO(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying
  $$A^T \cdot A = I, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$
  is a surface in $\mathbb{R}^{n^2}$
5. The \textit{pseudo-orthogonal} group $O(p,q,\mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices satisfying
\[ A^T \cdot \eta \cdot A = \eta, \quad \eta = \text{diag} \left(1, \ldots, 1, -1, \ldots, -1\right), \quad A \in \text{Mat}(n, \mathbb{R}) \]
is a surface in $\mathbb{R}^{n^2}$

6. The \textit{special pseudo-orthogonal} group $SO(p,q,\mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices $A \in \text{Mat}(n, \mathbb{R})$ satisfying
\[ A^T \cdot \eta \cdot A = \eta, \quad \det A = 1, \quad \eta = \text{diag} \left(1, \ldots, 1, -1, \ldots, -1\right) \]
is a surface in $\mathbb{R}^{n^2}$

7. The \textit{unitary} group $U(n)$ consisting of all $n \times n$ complex matrices satisfying
\[ A^\dagger \cdot A = \mathbb{I}, \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$

8. E.g. $U(1) \simeq S^1$

9. The \textit{special unitary} group $SU(n)$ consisting of all $n \times n$ complex matrices satisfying
\[ A^\dagger \cdot A = \mathbb{I}, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$

10. E.g. $SU(2) \simeq S^3$

11. The \textit{pseudo-unitary} group $U(p,q)$ consisting of all $n \times n$, $n = p+q$ complex matrices satisfying
\[ A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag} \left(1, \ldots, 1, -1, \ldots, -1\right), \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$
12. The special pseudo-unitary group $SU(p, q)$ consisting of all $n \times n$, $n = p + q$ complex matrices $A \in \text{Mat}(n, \mathbb{C})$ satisfying

$$A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag}(1, \ldots, 1, -1, \ldots, -1), \quad \det A = 1$$

is a surface in $\mathbb{R}^{2n^2}$

**Definition 2.1.7.** A manifold $G$ is called a Lie group if it has given on it a group operation with the property that the maps $\varphi : G \mapsto G$, defined by $\varphi(g) = g^{-1}$ (i.e. the taking of inverses) and $\psi : G \times G \mapsto G$ defined by $\psi(g, h) = gh$ (i.e. the group multiplication), are smooth maps.
9 Projective Spaces

9.1 Real Projective Space

Definition. The \textit{real projective space} $\mathbb{R}P^n$ is the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin. Equivalently it is the set of equivalence classes of nonzero vectors in $\mathbb{R}^{n+1}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.

Since each line passing through the origin intersect a sphere $S^n$ centred at the origin in exactly two points, the points of $\mathbb{R}P^n$ are in one-to-one correspondence with the pairs of diametrically opposite points of the $n$-sphere. We may think of $\mathbb{R}P^n$ as obtained from $S^n$ by gluing, that is identifying, diametrically opposite points. So, $\mathbb{R}P^n \cong S^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ maps a point of $S^n$ to the diametrically opposite point.

The projective line

$$\mathbb{R}P^1 \cong S^1/\mathbb{Z}_2 \cong S^1 \cong U(1)$$

$\mathbb{R}P^2$ is called the projective plane
9.2 Quaternions, $SU(2)$, $SO(3)$, $\mathbb{R}P^3$

**Definition.** The set $\mathbb{H}$ of *quaternions* consists of all linear combinations

$$q \in \mathbb{H}, \quad q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R},$$

and $1, i, j, k$ are linearly independent (so it is a 4-dim vector space).

We introduce the following multiplication in $\mathbb{H}$

$$i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k,$$

$$i \cdot i \equiv i^2 = -1, \quad j \cdot j \equiv j^2 = -1, \quad k \cdot k \equiv k^2 = -1,$$

$$i \cdot 1 = i = 1 \cdot i, \quad j \cdot 1 = j = 1 \cdot j, \quad k \cdot 1 = k = 1 \cdot k, \quad 1 \cdot 1 = 1,$$

which makes $\mathbb{H}$ an associative algebra over the field of real numbers.
For each quaternion
\[ q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R}, \]
we define
\[ A(q) = \begin{pmatrix} a + b i & c + d i \\ -c + d i & a - b i \end{pmatrix}, \quad A(q) \in \text{Mat}(2, \mathbb{C}). \]

**Lemma 1.** The map \( q \mapsto A(q) \) is one-to-one and
\[ A(q_1 q_2) = A(q_1)A(q_2) \]
so that this map is an algebra *monomorphism* which means it is an injective (one-to-one) homomorphism (consistent with multiplication).

Note
\[ A(\mathbf{i}) = i\sigma_3, \quad A(\mathbf{j}) = i\sigma_2, \quad A(\mathbf{k}) = i\sigma_1, \]
where \( \sigma_i \) are the Pauli matrices.
We define an operation of \textit{conjugation} on $\mathbb{H}$ by

$$\bar{q} = a \mathbf{1} - b \mathbf{i} - c \mathbf{j} - d \mathbf{k}, \quad a, b, c, d \in \mathbb{R}$$

**Lemma 2.** The map $q \mapsto \bar{q}$ is an anti-isomorphism of $\mathbb{H}$, i.e. it is linear, and

$$\bar{q_1 q_2} = \bar{q_2 q_1}$$

We define the \textit{norm} $|q| \geq 0$ of a quaternion by

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2, \quad a, b, c, d \in \mathbb{R}$$

Then, if $|q| \neq 0$ then

$$q \cdot \frac{\bar{q}}{|q|^2} = 1$$

and therefore each nonzero quaternion has multiplicative inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

**Lemma 3.** The algebra $\mathbb{H}$ of quaternions is a \textit{division} algebra, i.e. for each nonzero quaternion $q$ there exists a quaternion $q^{-1}$ inverse to it

$$qq^{-1} = 1 = q^{-1}q$$
We have
\[ |q|^2 = \det A(q) \implies |q_1q_2|^2 = |q_1|^2|q_2|^2 \]
Thus, the set of quaternions with norm 1 forms a group under multiplication. It is denoted by \( \mathbb{H}_1 \).

For \( q \in \mathbb{H}_1 \) we have \( q^{-1} = \bar{q} \).

If we regard \( \mathbb{H} \) as \( \mathbb{R}^4 \) then \( \mathbb{H}_1 \) is the hypersurface
\[ |q|^2 = a^2 + b^2 + c^2 + d^2 = 1 \implies \mathbb{H}_1 \cong S^3 \cong SU(2) \]
Let $\mathbb{H}_0$ denote the 3-dim Euclidean space consisting of all quaternions $x$ satisfying $\bar{x} = -x$, i.e. with the real part zero. Then

$$|x|^2 = x \cdot \bar{x} = -x^2 \geq 0$$

**Lemma 4.** If $|q|^2 = 1$, then the transformation defined by

$$\alpha_q : x \mapsto qxq^{-1}, \quad x \in \mathbb{H}_0$$

is a rotation of 3-dim Euclidean space $\mathbb{H}_0 = \mathbb{R}^3$.

Thus, the map $q \mapsto \alpha_q$ is a homomorphism from $\mathbb{H}_1 \cong SU(2)$ to the group of rotations of $\mathbb{R}^3$.

Obviously, $\alpha_q$ is the identity map when $q = \pm 1$.

Check that every rotation is of the form (9.6).

Thus, $SO(3)$ is isomorphic to $SU(2)/\{1, -1\}$.

Since $\{1, -1\} \cong Z_2$, where $Z_2$ identifies $A$ and $-A$ which correspond to diametrically opposite points of $S^3 \cong SU(2)$, we get

$$SO(3) \cong SU(2)/Z_2 \cong S^3/Z_2 \cong \mathbb{R}P^3$$

**Remark.** $\mathbb{H}_1 \cong SU(2)$ can be also used to prove that

$$SO(4) \cong (SU(2) \times SU(2))/Z_2$$
9.3 \( \mathbb{R}P^n \) as a manifold

Let’s introduce a manifold structure on \( \mathbb{R}P^n \).

\( \mathbb{R}P^n \) as the set of equivalence classes of nonzero vectors in \( \mathbb{R}^{n+1} \) with coordinates \( y^0, \ldots, y^n \).

For each \( q = 0, 1, \ldots, n \), let \( U_q \) denote the set of equivalence classes of vectors \( (y^0, \ldots, y^n) \) with \( y^q \neq 0 \). Obviously, \( \mathbb{R}P^n = \bigcup_{q=0}^{n} U_q \).

We introduce the following local coordinates on \( U_0 \)

\[
x^i_0 = \frac{y^i}{y^0}, \quad i = 1, \ldots, n.
\] (9.7)

Then, the local coordinates on each \( U_q \) are introduced through the following recursion relations

\[
x^i_{q+1} = \frac{x^i_q}{x^1_q}, \quad i = 1, \ldots, n - 1, \quad x^i_{q+1} = -\frac{1}{x^1_q}.
\] (9.8)

One gets

\[
x^1_q = \frac{x^2_{q-1}}{x^1_{q-1}} = \frac{x^3_{q-2} x^1_{q-2}}{x^2_{q-2} x^1_{q-2}} = \frac{x^3_{q-2}}{x^2_{q-2}} = \frac{x^4_{q-3} x^1_{q-3}}{x^3_{q-3} x^1_{q-3}} = \frac{x^4_{q-3}}{x^3_{q-3}} = \ldots
\] (9.9)

and therefore \( x^1_q \neq 0 \) on \( U_{q+1} \) and up to a sign \( x^i_{q+1} \) can be expressed as the ratio \( y^{n_i}/y^{q+1} \) where the index \( n_i \) depends on \( i \) and \( q \). Eqs (9.8) also provide transition functions on the intersection \( U_q \cap U_{q+1} \).
\[ x_{q+1}^i = \frac{x_{q+1}^{i+1}}{x_1}, \quad i = 1, \ldots, n - 1, \quad x_{q+1}^n = -\frac{1}{x_1}. \quad (9.10) \]

\[ x_1 = \frac{y^{q+1}}{y^q} \quad (9.11) \]

The Jacobian of these transition functions can be easily found

\[ J_{(x_q)\to(x_{q+1})} = \left( -\frac{1}{x_1} \right)^{n+1} \neq 0. \quad (9.12) \]

Since, the Jacobian of the transition functions for arbitrary \( q \) and \( p \) is

\[ J_{(x_q)\to(x_p)} = J_{(x_q)\to(x_{q+1})} J_{(x_{q+1})\to(x_{q+2})} \cdots J_{(x_{p-1})\to(x_p)} \]

\[ = \prod_{k=q}^{p-1} \left( -\frac{1}{x_1} \right)^{n+1} = \prod_{k=q}^{p-1} \left( -\frac{y^k}{y^{k+1}} \right)^{n+1} \]

\[ = \left( (-1)^{p-q} \frac{y^q}{y^p} \right)^{n+1} \neq 0, \quad (9.13) \]

and it is a smooth function on \( U_q \cap U_p \), we conclude that the real projective space \( \mathbb{R}P^n \) is a smooth manifold.

Moreover, \( \mathbb{R}P^n \) is oriented for odd \( n \).

\( \mathbb{R}P^2 \) is called the \textit{projective plane}, and \( U_0 \) is called the finite part of the projective plane.
9.4 Complex Projective Space

**Definition.** The *complex projective space* $\mathbb{C}P^n$ is the set of equivalence classes of nonzero vectors in $\mathbb{C}^{n+1}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.

The charts are defined as in the real case, making $\mathbb{C}P^n$ a $2n$-dim smooth manifold.

Consider the *complex projective line* $\mathbb{C}P^1$. Its points are equivalence classes of nonzero pairs

$$(z^0, z^1) \sim (\lambda z^0, \lambda z^1), \quad \lambda \neq 0, \lambda \in \mathbb{C}$$

Consider the complex function

$$w_0(z^0, z^1) = \frac{z^1}{z^0}$$

defined on $U_0 : z^0 \neq 0$ which covers all $\mathbb{C}P^1$ except the equivalence class of $(0, 1)$ which contains all nonzero pairs of the form $(0, z^1)$.

We define $w_0$ as taking the value $\infty$ at this point.

Then, via the function $w_0$, $\mathbb{C}P^1$ becomes identified with the *extended complex plane*, i.e. the union of the ordinary complex plane with a *point at infinity*.

**Theorem 2.2.1.** The complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-dim sphere $S^2$.

**Proof.** Blackboard

By this result, the extended complex plane is often called the Riemann sphere.
The general complex projective space $\mathbb{C}P^n$

From each equivalence class of $(n+1)$-dim vectors $(z^0, z^1, \ldots, z^n)$ we choose as a representative a vector with the unit norm

$$|z^0|^2 + |z^1|^2 + \cdots |z^n|^2 = 1$$

This equation defines the unit sphere $S^{2n+1}$ in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.

This representative is not unique because one can multiply all $z^i$ by $e^{i\varphi}$, $\varphi \in \mathbb{R}$, i.e. by complex numbers of modulus 1 which form the group $U(1)$.

Thus,

The complex projective space $\mathbb{C}P^n$ can be obtained from the unit sphere $S^{2n+1} = \{z | \sum_{i=0}^{n} |z^i|^2 = 1\}$, by identifying all points differing by a scalar factor of the form $e^{i\varphi}$.

This identification provides a map

$$S^{2n+1} \mapsto \mathbb{C}P^n \cong S^{2n+1}/U(1)$$

such that the pre-image of each point of $\mathbb{C}P^n$ is topologically equivalent to the circle $S^1$.

In particular, since $\mathbb{C}P^1 \cong S^2$, we get a map

$$S^3 \mapsto S^2, \quad (z^0, z^1) \mapsto w = \frac{z^1}{z^0}, \quad |z^0|^2 + |z^1|^2 = 1$$

This map is called the Hopf bundle, or fibration or map. $S^1$ is the fibre space embedded in $S^3$, and the Hopf map projects $S^3$ onto the base space $S^2$. 
10  Elements of the Theory of Lie Groups

10.1 Neighbourhood of the Identity Element of a Lie Group

Let $G$ be a Lie group, let the point $g_0 = 1 \in G$ be the identity element of $G$, and let $T = T(1)$ be the tangent space at the identity element.

Let us express the group operations on $G$ in a chart $U_0$ containing $g_0$ in terms of local coordinates.

We choose coordinates in $U_0$ so that the identity element is the origin

$$1 = g_0 = (0, \ldots, 0)$$

Let

$$g_1 = (x^1, \ldots, x^n), \quad g_2 = (y^1, \ldots, y^n), \quad g_3 = (z^1, \ldots, z^n)$$

where $g_k$ are such that

$$g_k, g_k^{-1}, g_k g_l, g_k^{-1} g_l^{-1}, g_k g_l^{-1}, g_k^{-1} g_l \in U_0, \quad k, l = 1, 2, 3$$

Then,

$$g_1 g_2 = (\psi^1(x, y), \psi^2(x, y), \ldots, \psi^n(x, y)) = (\psi^i(x, y)), \quad i = 1, \ldots, n$$

$$\psi^i(x, y) = \psi^i(x^1, \ldots, x^n, y^1, \ldots, y^n), \quad i = 1, \ldots, n$$

$$g_1^{-1} = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^n(x)) = (\varphi^i(x)), \quad i = 1, \ldots, n$$

$$\varphi^i(x) = \varphi^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n$$

are the coordinates of $g_1 g_2$ and $g_1^{-1}$.

$\psi(x, y)$ and $\varphi(x)$ satisfy ( $i = 1, \ldots, n$)

1. $\psi^i(x, 0) = \psi^i(0, x) = x^i$
2. $\psi^i(x, \varphi(x)) = 0$
3. $\psi^i(x, \psi(y, z)) = \psi^i(\psi(x, y), z)$
Let $\psi(x, y)$ be sufficiently smooth. Then

$$
\psi^i(x, y) = x^i + y^i + b_{jk}^i x^j y^k + (\text{terms of order}) \geq 3
$$

$$
b_{jk}^i = \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \bigg|_{x=y=0}
$$

Let $\xi, \eta \in T$, and their components in terms of $x^i$ are $\xi^i$ and $\eta^i$.

**Definition.** The *commutator* $[\xi, \eta] \in T$ is defined by

$$
[\xi, \eta]^i = c_{jk}^i \xi^j \eta^k, \quad c_{jk}^i \equiv b_{jk}^i - b_{kj}^i
$$

It has 3 basic properties

1. $[\ , \ ]$ is a bilinear operation on the $n$-dim vector space $T$
2. Skew-symmetry: $[\xi, \eta] = -[\eta, \xi]$
3. Jacoby’s identity: $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$

**Proof of 3.** Blackboard
**Definition.** A *Lie algebra* is a vector space $\mathcal{G}$ over a field $F$ with a bilinear operation $[\cdot, \cdot]: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ which is called a commutator or a Lie bracket, such that the following axioms are satisfied:

- It is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in \mathcal{G}$
- It satisfies the Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Thus, the tangent space of a Lie group $G$ at the identity is with respect to the commutator operation a Lie algebra called the *Lie algebra of the Lie group* $G$.

Let $e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n}$ be the standard basis vectors of $T$ in terms of the coordinates $x^1, \ldots, x^n$.

Let us multiply $[\xi, \eta]^i = c^i_{jk} \xi^j \eta^k$ by $e_i$ and take the sum over $i$

$$[\xi, \eta] = c^i_{jk} \xi^j \eta^k e_i$$

Let us choose $\xi = e_j, \eta = e_k$. Taking into account that the components of the standard basis vector are

$$\left(e_m^i\right)^n = \delta^n_m$$

we get

$$[e_j, e_k] = c^i_{jk} e_i$$

**Definition.** The constants $c^i_{jk}$ which determine the commutation operation on a Lie algebra, and which are skew-symmetric in $j, k$, are called the *structure constants* of the Lie algebra.
10.2 One-parameter subgroups and canonical coordinates

**Definition.** A one-parameter subgroup of a Lie group $G$ is defined to be a parametric curve $F(t)$ on the manifold $G$ such that

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F(t)^{-1}$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = \frac{dF(t + \epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}(F(t)F(\epsilon))\bigg|_{\epsilon=0} = F(t)\frac{dF(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

Hence,

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F(t)^{-1}\dot{F}(t) = \dot{F}(0),$$

i.e. the induced action of left multiplication by $F(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0) = \text{const} \in T$.

Conversely, $\forall A \in T$, the equation

$$F(t)^{-1}\dot{F}(t) = A$$

is satisfied by a unique one-parameter subgroup $F(t)$ of $G$.

If $G$ is a matrix group then $F(t) = \exp At$.

We will use this notation for arbitrary Lie groups.
Let’s discuss how $F(t)$ and the push-forward map look like in a neighbourhood of $U_0$.  

Let $F(t) \in U_0$ have local coordinates $f^1(t), \ldots, f^n(t)$. Since $F(t)$ is a one-parameter subgroup the functions $f^i(t)$ satisfy

$$f^i(0) = 0, \quad f^i(t_1 + t_2) = \psi^i(f(t_1), f(t_2)), \quad f^i(-t) = \varphi^i(f(t))$$

Consider the map $G \mapsto G$ given by the left multiplication by $F(t)$

$$x \mapsto y = F(t) x, \quad x, y \in G$$

If the local coordinates of $x$ and $y$ are $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ then the left multiplication takes the form

$$x \mapsto y : y^i = \psi^i(f(t), x)$$

The corresponding push-forward map induced by the left multiplication is

$$F^*_\epsilon(t) : \xi^i \mapsto \eta^i = \frac{\partial \psi^i(f(t), x)}{\partial x^j} \xi^j, \quad \xi \in T_x G, \quad \eta \in T_y G$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = (\dot{f}^1(t), \ldots, \dot{f}^n(t)),$$

where

$$\dot{f}^i(t) = \frac{df^i(t + \epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d\psi^i(f(t), f(\epsilon))}{d\epsilon}\bigg|_{\epsilon=0} = \frac{\partial \psi^i(f(t), x)}{\partial x^j}\bigg|_{x=f(t)} \cdot \dot{f}^j(0)$$

Thus, $F^*_\epsilon(t)$ sends $\dot{F}(0)$ to $\dot{F}(t)$. Similarly, we get

$$\dot{f}^i(0) = \frac{df^i(-t + \epsilon)}{d\epsilon}\bigg|_{\epsilon=t} = \frac{d\psi^i(f(-t), f(\epsilon))}{d\epsilon}\bigg|_{\epsilon=t} = \frac{\partial \psi^i(f(t), x)}{\partial x^j}\bigg|_{x=f(t)} \cdot \dot{f}^j(t)$$

and therefore $F^*_\epsilon(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0)$. 

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**Definition.** For each $h \in G$ the transformation $G \mapsto G$ defined by $g \mapsto hgh^{-1}$ is called the *inner automorphism* of $G$ determined by $h$.

Any inner automorphism does not move $g_0$, $hg_0h^{-1} = g_0$, and therefore the push-forward (induced linear) map of the tangent space $T$ to $G$ at $g_0$ is a linear transformation of $T$ denoted by (or by $\text{Adj}$)

$$\text{Ad}_h : T \mapsto T$$

It satisfies

1. $\text{Ad}_{g_0} = id$ where $id$ is the identity transformation of $T$
2. $\text{Ad}_{h_1}\text{Ad}_{h_2} = \text{Ad}_{h_1h_2}$ for all $h_1, h_2 \in G$
   because $h_1h_2gh_2^{-1}h_1^{-1} = (h_1h_2)g(h_1h_2)^{-1}$
3. Choosing $h_1 = h, h_2 = h^{-1}$, we get $\text{Ad}_{h^{-1}} = \text{Ad}_h^{-1}$

This means that the map $h \mapsto \text{Ad}_h$ is a *linear representation* of the group $G$, i.e. a homomorphism to a group of linear transformations

$$\text{Ad} : G \mapsto GL(n, \mathbb{R}), \quad h \mapsto \text{Ad}_h = \text{Ad}(h)$$

where $n = \dim (G)$.

This representation of $G$ is called *adjoint*.

For commutative Lie groups $G$, e.g. $U(1)$, the adjoint representation $\text{Ad}$ is trivial, i.e. $\text{Ad}_h = 1 \ \forall h \in G$. 
In terms of local coordinates in a neighbourhood of $U_0$ we get the following. Let us denote the inner automorphism of $G$ determined by $h$ by $AD(h)$

$$AD(h): g \mapsto hgh^{-1}, \quad g, h \in G$$

The corresponding push-forward map is

$$AD(h)_*: \xi^i \mapsto \eta^i = \frac{\partial \psi^i(\psi(h, x), \varphi(h))}{\partial x^j} \xi^j, \quad \xi \in T_g G, \quad \eta \in T_{hgh^{-1}} G$$

where $g$ has local coordinates $(x^1, \ldots, x^n)$.

If $x = 0$ then $g = g_0$ and both $\xi, \eta \in T_{g_0} G = T$, and we get $Ad_h$

$$Ad_h: \xi^i \mapsto \eta^i = \frac{\partial \psi^i(\psi(h, x), \varphi(h))}{\partial x^j} \bigg|_{x=0} \xi^j = \frac{\partial \psi^i(z, \varphi(h))}{\partial z^k} \bigg|_{z=h} \frac{\partial \psi^k(h, x)}{\partial x^j} \bigg|_{x=0} \xi^j. \quad (10.14)$$

This formula can be used to show in particular that

$$Ad_{h_1}Ad_{h_2} = Ad_{h_1h_2}$$
Let \( F(t) = \exp At \) be a one-parameter subgroup of a Lie group \( G \).
Then, \( \text{Ad}_{F(t)} \) is a one-parameter subgroup of \( GL(n, \mathbb{R}) \), and the vector \( \frac{d}{dt}\text{Ad}_{F(t)}|_{t=0} \) lies in the Lie algebra \( gl(n, \mathbb{R}) \cong Mat(n, \mathbb{R}) \) of the group \( GL(n, \mathbb{R}) \) and can be regarded as a linear operator.

This operator is denoted by \( \text{ad}_A \) and is given by
\[
\text{ad}_A : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad B \mapsto [A, B], \quad B \in T \cong \mathbb{R}^n
\]
The formula is obtained by using (10.14) where we replace \( \xi \rightarrow B, \) \( h \rightarrow f(t), \) differentiate with respect to \( t, \) set \( t = 0 \) and use that \( \dot{f}^i(0) = A^i \) (prove the formula for \( \text{ad}_A \)).

One-parameter subgroups can be used to define so-called canonical coordinates in a neighbourhood of the identity of a Lie group \( G \).

Let \( A_1, \ldots, A_n \) form a basis for the Lie algebra \( T \).
\( \forall A = \sum_i A_i x^i \in T \exists \) a one-parameter group \( F(t) = \exp At. \)

To the point \( F(1) = \exp A \) we assign as coordinates the coefficients \( x^1, \ldots, x^n \) which gives us a system of coordinates in a sufficiently small neighbourhood of \( g_0 = 1 \in G. \)

These are called the canonical coordinates of the first kind.
Another system of coordinates is obtained by introducing 
\( F_i(t) = \exp A_i t \) and representing a point \( g \) sufficiently close to \( g_0 \) as 
\[
g = F_1(t_1)F_2(t_2) \cdots F_n(t_n)
\]
for small \( t_1, \ldots, t_n \).
Assigning coordinates \( x^1 = t_1, \ldots x^n = t_n \) to the point \( g \), we get the \textit{canonical coordinates of the second kind}.

\textbf{Theorem 3.1.1.} If the functions \( \psi^i(x, y) \) defining the multiplication of points \( x, y \) of a Lie group \( G \) are real analytic (representable by power series) then in some neighbourhood of \( g_0 \in G \) the structure of the Lie algebra of \( G \) determines the multiplication in \( G \).

\textbf{Proof.} See the textbook.
Definition 3.1.3. A Lie algebra $\mathcal{G} = \{\mathbb{R}^n, c^{i}_{jk}\}$ is said to be simple if it is noncommutative and has no proper ideals, i.e. subspaces $\mathcal{I} \neq \mathcal{G}, 0$ for which $[\mathcal{I}, \mathcal{G}] \subset \mathcal{I}$, and semisimple if $\mathcal{G} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_k$ where the $\mathcal{I}_j$ are ideals which are simple as Lie algebras.

These ideals are pairwise commuting $[\mathcal{I}_i, \mathcal{I}_j] = 0$ for $i \neq j$.

A Lie group is defined to be simple or semisimple according to its Lie algebra.

Definition. The Killing form on an arbitrary Lie algebra $\mathcal{G}$ is defined by

$$\langle A, B \rangle = -\text{tr}(\text{ad}_A \text{ad}_B)$$

Theorem 3.1.4.

(i) If the Lie algebra $\mathcal{G}$ of a Lie group $G$ is simple, then the linear representation $\text{Ad} : G \mapsto GL(n, \mathbb{R})$ is irreducible, i.e. $\mathcal{G}$ has no proper invariant subspaces under the group of inner automorphisms $\text{Ad}_G$.

(ii) If the Killing form of a Lie algebra is positive definite then the Lie algebra is semisimple.

Proof. Blackboard

Remark. A stronger result due to Killing and Cartan is

A Lie algebra is semisimple if and only if its Killing form is non-degenerate.
10.3 Linear representations

Definition 3.2.1.

(i) A **linear representation** of a group $G$ of dim $= n$ is a homomorphism

$$\rho : G \mapsto GL(r, \mathbb{R}) \quad \text{or} \quad \rho : G \mapsto GL(r, \mathbb{C})$$

from $G$ to a group of real or complex matrices.

(ii) Given a representation $\rho$ of $G$, the map

$$\chi_\rho : G \mapsto \mathbb{R} \quad \text{or} \quad G \mapsto \mathbb{C}$$

defined by

$$\chi_\rho(g) = \text{tr} \rho(g), \quad g \in G$$

is called the **character** of the representation $\rho$.

(iii) A representation $\rho$ of $G$ is said to be **irreducible** if the vector space $\mathbb{R}^r$ (or $\mathbb{C}^r$) contains no proper subspaces invariant under the matrix group $\rho(G)$. 
Theorem 3.2.2 (Schur’s Lemma).

Let
\[ \rho_i : G \mapsto GL(r_i, \mathbb{R}), \quad i = 1, 2 \]
be two irreps of a group \( G \). If \( A : \mathbb{R}^{r_1} \mapsto \mathbb{R}^{r_2} \) is a linear transformation changing \( \rho_1 \) into \( \rho_2 \), i.e. satisfying
\[ A\rho_1(g) = \rho_2(g)A, \quad \forall g \in G \]
then either \( A \) is the zero transformation or else a bijection (in which case \( r_1 = r_2 \)).

Proof. Blackboard
If $G$ is a Lie group and a representation $\rho : G \mapsto GL(r, \mathbb{R})$ is a smooth map, then the push-forward map $\rho_*$ is a linear map from the Lie algebra $\mathcal{G} = T(1)$ to the space of all $r \times r$ matrices

$$\rho_* : \mathcal{G} \mapsto \text{Mat}(r, \mathbb{R})$$

Verify that $\rho_*$ is a representation of the Lie algebra $\mathcal{G}$, i.e. that it is a Lie algebra homomorphism:

1. It is linear
2. It preserves commutators

$$\rho_*[\xi, \eta] = [\rho_*\xi, \rho_*\eta]$$
**Definition.** A representation

\[ \rho : G \mapsto GL(r, \mathbb{R}) \] or \[ \rho : G \mapsto GL(r, \mathbb{C}) \]

is called *faithful* if it is one-to-one, i.e. if its kernel is trivial

\[ \rho(g) \neq \mathbb{I} \quad \text{unless} \quad g = g_0 \]

If a Lie group has a faithful representation then it can be realised as a matrix Lie group.

Any matrix Lie group obviously has a faithful representation.

However, not every Lie group can be realised as a matrix Lie group.

One such an example is the group \( \widetilde{SL}(2, \mathbb{R}) \) of all transformations of the real line of the form

\[ x \rightarrow x + 2\pi a + \frac{1}{i} \ln \frac{1 - ze^{-ix}}{1 - \bar{z}e^{ix}}, \]

where \( x \in \mathbb{R}, a \in \mathbb{R}, z \in \mathbb{C}, |z| < 1 \) and \( \ln \) is the main branch of the natural logarithmic function, i.e. the continuous branch determined by \( \ln 1 = 0 \).

\( \widetilde{SL}(2, \mathbb{R}) \) is a universal covering group of \( SL(2, \mathbb{R}) \), i.e. it has the same Lie algebra and it is simply connected.
11 Homogeneous Spaces

11.1 Action of a group on a manifold

Definition 5.1.1. We say that a Lie group $G$ is represented as a *group of transformations* of a manifold $M$, or has a *left action* on $M$ if

1. there is associated with each of its elements $g$ a diffeomorphism from $M$ to itself
   \[ x \mapsto T_g(x), \quad x \in M, \]
   such that $T_{gh} = T_g T_h$ for all $g, h \in G$

2. $T_g(x)$ depends smoothly on the arguments $g, x$, i.e. the map $(g, x) \mapsto T_g(x)$ is a smooth map from $G \times M$ to $M$.

The Lie group $G$ is said to have a *right action* on $M$ if the above definition is valid with $T_{gh} = T_g T_h$ replaced by $T_g T_h = T_{hg}$.

**Example 1.** Let $M = G$. Is the action below left or right?

1. $h \mapsto T_g(h) = gh, \quad h \in G$
2. $h \mapsto T_g(h) = hg, \quad h \in G$
3. $h \mapsto T_g(h) = g^{-1}h, \quad h \in G$
4. $h \mapsto T_g(h) = hg^{-1}, \quad h \in G$

**Example 2.** Any group of real $n \times n$ matrices acts on $\mathbb{R}^n$, e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$
**Definition.** The action of a group $G$ on $M$ is said to be *transitive* if for every two points $x, y$ of $M$ there exists an element $g$ of $G$ such that $T_g(x) = y$.

**Definition 5.1.2.** A manifold on which a Lie group acts transitively is called a *homogeneous* space of the Lie group.

In particular, $G$ is a homogeneous space for itself, e.g. as $h \mapsto T_g(h) = gh, \ h \in G$.

$G$ is called the *principal* homogeneous space.

**Definition.** Let $x$ be any point of a homogeneous space $M$ of a Lie group $G$. The *isotropy* group (or *stationary* group) $H_x$ of the point $x$ is the stabiliser of $x$ under the action of $G$:

$$H_x = \{ h | T_h(x) = x \}.$$

**Lemma 5.1.3.** All isotropy groups $H_x$ of points $x$ of a homogeneous space are isomorphic.

**Proof.** Let $x, y$ be any two points of the homogeneous space and $g$ be an element of the Lie group such that $T_g(x) = y$. The map $H_x \mapsto H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism.

*Prove that $ghg^{-1}$ is an element of $H_y$.*
**Theorem 5.1.4.** There is a one-to-one correspondence between the points of a homogeneous space $M$ of a group $G$, and the left cosets $gH$ of $H$ in $G$, where $H$ is the isotropy group and $G$ acts on the left.

**Proof.** Recall that if $G$ is a group, and $H$ is a subgroup of $G$, and $g \in G$, then $gH = \{gh : h \in H\}$ is the left coset of $H$ in $G$ with respect to $g$, while $Hg = \{hg : h \in H\}$ is the right coset.

Let $x_0$ be any point of $M$. Then, we put in correspondence to each left coset $gHx_0$ the point $T_g(x_0) \in M$. This correspondence is independent of the choice of representative of the coset, one-to-one, and onto. 

_Prove it_

Thus, we can write

$$M \cong G/H$$
11.2 Examples of Homogeneous Spaces

1. Sphere:
\[ S^n \cong O(n + 1)/O(n) \cong SO(n + 1)/SO(n) \]

2. Real projective space:
\[ \mathbb{R}P^n \cong O(n + 1)/(O(1) \times O(n)) \]

3. Torus:
\[ T^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/Z^n \]

4. Stiefel manifolds:
\[ V_{n,k} \cong O(n)/O(n - k) \cong SO(n)/SO(n - k) \]

5. Real Grassmanian manifolds:
\[ G_{n,k} \cong O(n)/(O(k) \times O(n - k)) \]

6. Homogeneous spaces for \( U(n) \)
   
   (a) Sphere:
\[ S^{2n+1} \cong U(n + 1)/U(n) \cong SU(n + 1)/SU(n) \]

   (b) Complex projective space:
\[ \mathbb{C}P^n \cong U(n + 1)/(U(1) \times U(n)) \]

   (c) Complex Grassmanian manifolds:
\[ G_{n,k}^\mathbb{C} \cong U(n)/(U(k) \times U(n - k)) \]
12 Vector Bundles on a Manifold

12.1 Tangent bundle $T(M)$

**Definition.** The *tangent bundle* $T(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T(M)$ are the pairs $(x, \xi)$, $x \in M$ and $\xi \in T_xM$

2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U^T_q$ of $T(M)$ is the set of all pairs $(x, \xi)$ where

   $$x = (x^1_q, \ldots, x^n_q) \in U_q \quad \text{and} \quad \xi = \xi^i_q \frac{\partial}{\partial x^i_q} \in T_xM$$

   with the local coordinates

   $$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, \xi^1_q, \ldots, \xi^n_q) = (x^i_q, \xi^i_q)$$

**Proposition 7.1.1.** The tangent bundle $T(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U^T_q \cap U^T_p$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, \xi^i_p) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^i_p}{\partial x^k_q} \xi^k_p)$$

The Jacobian matrix is

$$\left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = \begin{pmatrix} A & 0 \\ H & A \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\partial x^i_p}{\partial x^j_q} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{\partial^2 x^i_p}{\partial x^j_q \partial x^k_q} \xi^k_p \end{pmatrix}$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = (\det A)^2 > 0$$
12.2 Cotangent bundle $T^*(M)$

**Definition.** The *cotangent bundle* $T^*(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T^*(M)$ are the pairs $(x, p), x \in M,$ and $p$ is a co-vector at the point $x$: $p \in T^*_x M$

2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U^{T^*}_q$ of $T^*(M)$ is the set of all pairs $(x, p)$ where
   
   $x = (x^1_q, \ldots, x^n_q) \in U_q$ and $p = p_q dx^i_q \in T^*_x M$

with the local coordinates

$$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, p_{q1}, \ldots, p_{qn}) = (x^i_q, p_{qi})$$

**Proposition.** The cotangent bundle $T^*(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U^{T^*}_q \cap U^{T^*}_p$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, p_{pi}) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^i_q}{\partial x^j_p} p_{qk})$$

The Jacobian matrix is

$$\left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = \begin{pmatrix} A & 0 \\ \tilde{H} & A^{-1} \end{pmatrix}, \quad A = \left( \frac{\partial x^i_p}{\partial x^j_q} \right), \quad \tilde{H} = \left( \frac{\partial^2 x^k_q}{\partial x^j_q \partial x^i_p} p_{qk} \right)$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = 1 > 0$$
The existence of a metric $g_{ij}$ on $M$ gives rise to a map

$$T(M) \mapsto T^*(M) : (x^i, \xi^i) \mapsto (x^i, g_{ij} \xi^j)$$

Since $\omega = p_i dx^i$, a differential one-form on $M$, is invariant under a change of coordinates of $T^*(M)$, it is a differential form on $T^*(M)$. Its differential

$$\Omega = d\omega = dp_i \wedge dx^i$$

is a nondegenerate closed, $d\Omega = 0$, 2-form on $T^*(M)$.

Thus, $T^*(M)$ is a symplectic manifold, i.e. it is equipped with a closed nondegenerate 2-form.
12.3 Normal vector bundle on a submanifold

Let $M$ be an $n$-dim Riemann manifold with metric $g_{ij}$, and let $N$ be a smooth $k$-dim submanifold of $M$. We assume that $N$ is defined by a non-singular system of $(n - k)$ equations.

Recall that the scalar product of two vectors $\xi, \eta \in T_x M$ is given by the metric

$$\langle \xi, \eta \rangle = g_{ij} \xi^i \eta^j$$

Let $x \in N$, and let $\nu \in T_x M$, and let $\nu$ be orthogonal to $N$ at $x$, i.e. orthogonal to the tangent space to $N$ at $x$, which is a $k$-dim subspace of $T_x M$. So, $\nu$ form a $(n - k)$-dim subspace of $T_x M$.

Definition and Theorem. The normal vector bundle $\nu_M(N)$ on the submanifold $N$ in $M$ is an $n$-dim submanifold of $T(M)$ defined as

1. The points of $\nu_M(N)$ are the pairs $(x, \nu)$, $x \in N$, $\nu \in T_x M$, $\nu \perp N$

2. Given a chart $U$ of $M$ with suitable local coordinates

   $y^i, i = 1, \ldots, n$, $N$ is defined in $U$ by the equations

   $y^{k+1} = 0, \ldots y^n = 0$,

   and $y^1, \ldots, y^k$ serve as local coordinates on $N$.

3. The normal bundle $\nu_M(N)$ is determined as an $n$-dim submanifold of $T(M)$ by the equations

   $y^{k+1} = 0, \ldots y^n = 0$, \quad g_{ij} \nu^j = 0, \quad i = 1, \ldots, k$
Examples.

1. Let $M = \mathbb{R}^n$, and let $N$ be defined by the nonsingular system

$$f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y = (y^1, \ldots, y^n)$$

where $y^i$ are Euclidean coordinates on $\mathbb{R}^n$.

Then, the vectors $\vec{\nabla} f_1, \ldots, \vec{\nabla} f_{n-k}$, are at each point of $N$ perpendicular to $N$ and linearly independent.

Hence,

$$\nu_{\mathbb{R}^n}(N) \cong N \times \mathbb{R}^{n-k}$$
2. More generally, if \( N \) is defined as a submanifold of \( M \) by a non-singular system

\[
    f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y \in M
\]

then at each point \( x \in N \) the vector fields

\[
    e_a^i(x) = g^{jk}(y) \frac{\partial f_a}{\partial y^k} \bigg|_{y=y(x)}, \quad g^{jk}(y) g_{kj}(y) = \delta^i_j, \quad a = 1, \ldots, n-k
\]

are linear independent, and for each \( \xi \in T_xN \) they satisfy

\[
    g_{ij} e_a^i(x) \xi^j(x) = \frac{\partial f_a}{\partial y^j} \xi^j(x) \bigg|_{y=y(x)} = 0
\]

Thus, they are orthogonal to \( N \) and any vector normal to \( N \) at \( x \in N \) has the form

\[
    \nu = \nu^a e_a(x)
\]

The correspondence

\[
    (x, \nu) \mapsto (x, \nu^1, \ldots, \nu^{n-k})
\]

is then a diffeomorphism

\[
    \nu_M(N) \cong N \times \mathbb{R}^{n-k}
\]
3. In particular, if $N$ is the boundary of $\partial A$ of a manifold $A$ with boundary defined by an inequality $f(y) \leq 0$ then $\partial A$ is defined by the single equation $f(y) = 0$, and the normal bundle to the boundary decomposes as a direct product

$$\nu_M(\partial A) \cong \partial A \times \mathbb{R}$$

The bundles we considered are particular cases of *smooth fibre bundles*, see the textbook.