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Differentiable Manifolds

1 Definition of Manifold

$\mathbb{R}^n$ is an $n$-dim Cartesian or vector or linear space.

**Definition.** A *region*, or a region without boundary, (“open set”) is a set $D$ of points in $\mathbb{R}^n$ such that together with each point $P_0$, $D$ also contains all points sufficiently close to $P_0$, i.e.

$$\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 :$$

all points $P = (x^1, \ldots, x^n)$ satisfying $|x^i - x_0^i| < \epsilon$, $i = 1, \ldots, n$ also lie in $D$. 
\( \mathbb{R}^n \) is an \( n \)-dim Cartesian or vector or linear space.

**Definition.** A region, or a region without boundary, ("open set") is a set \( D \) of points in \( \mathbb{R}^n \) such that together with each point \( P_0 \), \( D \) also contains all points sufficiently close to \( P_0 \), i.e.

\[
\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 : \forall P = (x^1, \ldots, x^n) \text{ satisfying } |x^i - x_0^i| < \epsilon, i = 1, \ldots, n \text{ also lie in } D.
\]

**Definition.** A region with boundary is obtained from a region \( D \) by adjoining all boundary points (i.e. points not in \( D \), yet having points of \( D \) arbitrarily close to them). The boundary of a region is just the set of boundary points.
\( \mathbb{R}^n \) is an \( n \)-dim Cartesian or vector or linear space.

**Definition.** A region, or a region without boundary, ("open set") is a set \( D \) of points in \( \mathbb{R}^n \) such that together with each point \( P_0 \), \( D \) also contains all points sufficiently close to \( P_0 \), i.e.

\[
\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 : \\
\text{all points } P = (x^1, \ldots, x^n) \text{ satisfying } |x^i - x_0^i| < \epsilon, i = 1, \ldots, n \text{ also lie in } D.
\]

**Definition.** A region with boundary is obtained from a region \( D \) by adjoining all boundary points (i.e. points not in \( D \), yet having points of \( D \) arbitrarily close to them). The boundary of a region is just the set of boundary points.

**Definition.** \( n \)-dim *Euclidean* space is \( \mathbb{R}^n \) with the distance \( l \) between any two points given by

\[
l^2 = \sum_{i=1}^{n} (x^i - y^i)^2
\]
Definition 1.1.1. A differentiable $n$-dimensional manifold is a set $M$ (whose elements we call “points”) together with the following structure on it. The set $M$ is the union of a finite or countably infinite collection of subsets $U_q$ with the following properties

(i) Each subset $U_q$ has defined on it coordinates $x^\alpha_q, \alpha = 1, \ldots, n$ (called local coordinates) by virtue of which $U_q$ is identifiable with a region of Euclidean $n$-space $\mathbb{R}^n$ with Euclidean coordinates $x^\alpha_q$. The $U_q$ with their coordinate systems are called charts or local coordinate neighbourhoods.

(ii) Each non-empty intersection $U_p \cap U_q$ of a pair of charts thus has defined on it two coordinate systems, the restrictions of $(x^\alpha_p)$ and $(x^\alpha_q)$. It is required that under each of these coordinatisations the intersection $U_p \cap U_q$ is identifiable with a region of $\mathbb{R}^n$, and that each of these coordinate systems be expressible in terms of the other in a one-to-one differentiable manner. Thus, if the transition functions from $x^\alpha_q$ to $x^\alpha_p$ and back are given by

$$
x^\alpha_p = x^\alpha_p(x^1_q, \ldots, x^n_q), \quad \alpha = 1, \ldots, n, \\
x^\alpha_q = x^\alpha_q(x^1_p, \ldots, x^n_p), \quad \alpha = 1, \ldots, n,
$$

(1.1)

then in particular the Jacobian $\det(\partial x^\alpha_p / \partial x^\beta_q)$ is nonzero on $U_p \cap U_q$.

The general smoothness class of the transition functions for all intersecting pairs $U_p, U_q$ is called the smoothness class of the manifold $M$ with its accompanying atlas of charts $U_q$. 

Example 1. Any Euclidean space of regions is a manifold.

Example 2. A region of complex space $\mathbb{C}^n$ can be regarded as a region of $\mathbb{R}^{2n} \Rightarrow \mathbb{C}^n$ is a manifold.

Example 3. A 2-sphere $S^2$ is a manifold.

Example 4. A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.
**Example 1.** Any Euclidean space of regions is a manifold.

**Example 2.** A region of complex space $\mathbb{C}^n$ can be regarded as a region of $\mathbb{R}^{2n} \implies \mathbb{C}^n$ is a manifold.

**Example 3.** A 2-sphere $S^2$ is a manifold.

**Example 4.** A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.

**Example 5.** Given two manifolds $M = \bigcup_q U_q$, $N = \bigcup_p V_p$ we construct their *direct product* $M \times N$ as follows:

The points of the manifold $M \times N$ are ordered pairs $(m, n)$, and covering by local coordinate neighbourhoods is given by

$$M \times N = \bigcup_{q,p} U_q \times V_p$$

where if $x_q^\alpha$ are the coordinates on $U_q$ and $y_p^\beta$ on $V_p$ then the coordinates on $U_q \times V_p$ are $(x_q^\alpha, y_p^\beta)$.

E.g. $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times S^1$, $S^1 \times \mathbb{R}$, $S^1 \times S^1$, $\mathbb{R}^m \times \mathbb{R}^n$. 
2 Elements of Topology

The definition of manifold is very general. To restrict it we need some basic concepts of topology.

**Definition.** A topological space is a set $X$ (of “points”) of which certain subsets, called the *open sets* of the topological space, are distinguished. These open sets have to satisfy:

(a) the intersection of any two (and hence of any finite collection) of them should again be an open set;

(b) the union of any collection of open sets must again be open;

(c) the empty set and the whole set $X$ must be open.

The complement of any open set is called a *closed set* of the topological space.

In Euclidean space $\mathbb{R}^n$ the “Euclidean topology” is the usual one where the open sets are the open regions.
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In Euclidean space $\mathbb{R}^n$ the “Euclidean topology” is the usual one where the open sets are the open regions.

**Definition.** Given any subset $A \in \mathbb{R}^n$, the *induced topology* on $A$ is that with open sets the intersections $A \cap U$ where $U$ ranges over all open sets of $\mathbb{R}^n$.

**Definition.** A map $f : X \mapsto Y$ of one topological space to another is *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subseteq Y$ is open in $X$.

**Definition.** Two topological spaces are *topologically equivalent* or *homeomorphic* if there is a one-to-one and onto map between them such that both it and its inverse are continuous.
**Definition 1.1.2.** The topology on a manifold $M$ is given by the following specification of the open sets.

In every local coordinate neighbourhood $U_q$ the open regions are to be open in the topology on $M$; the totality of open sets of $M$ is then obtained by admitting as open also arbitrary unions of countable collections of such regions, i.e. by closing under countable unions.

With this topology the continuous maps of a manifold $M$ turn out to be those which are continuous in the usual sense on each local coordinate neighbourhood $U_q$.

Any open subset $V$ of $M$ inherits, i.e. has induced on it, the structure of a manifold: $V = \bigcup_q V_q$ where $V_q = V \cap U_q$. 
**Definition.** A *metric space* is a set which comes equipped with a “distance function”, i.e. a real-valued function $\rho(x, y)$ defined on pairs $x, y$ of its elements (“points”), and having the following properties

(i) Symmetry: $\rho(x, y) = \rho(y, x)$,

(ii) Positivity: $\rho(x, x) = 0$, $\rho(y, x) > 0$ if $x \neq y$,

(iii) The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$,

**Example.** $n$-dim Euclidean space is a metric space with

$$
\rho(x, y) = \sqrt{\sum_{\alpha=1}^{n} (x^\alpha - y^\alpha)^2}.
$$

A metric space is topologised by taking as its open sets the unions of arbitrary collections of “open balls” where by *open ball* with centre $x_0$ and radius $\epsilon$ we mean the set of all points $x$ satisfying $\rho(x, x_0) < \epsilon$. 
**Definition.** A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius \( \rho(x, y)/3 \) with centres at \( x, y \) do not intersect.

All topological spaces we consider will be Hausdorff.

In particular our manifolds will be Hausdorff spaces.
**Definition.** A topological space $X$ is said to be *compact* if every countable collection of open sets covering $X$ contains a finite subcollection already covering $X$.

If $X$ is a metric space then compactness is equivalent to the condition that from every sequence of points of $X$ a convergent subsequence can be selected.

**Definition.** A topological space is connected if any two points can be joined by a continuous path.
**Definition 1.1.3.** A manifold $M$ is said to be *oriented* if for every pair $U_p, U_q$ of intersecting local coordinate neighbourhoods the Jacobian $J_{pq} = \det(\partial x_\alpha^p/\partial x_\beta^q)$ of the transition functions is positive.

Euclidean $\mathbb{R}^n$ and $S^2$ are oriented.

**Definition 1.1.4.** We say that the coordinate systems $x$ and $y$ define the *same orientation* of $\mathbb{R}^n$ if $J > 0$ and *opposite orientations* if $J < 0$.

Euclidean $\mathbb{R}^n$ possesses two possible orientations, and any *connected* oriented manifold has exactly two orientations.
3 Mappings of Manifolds. Tensors on Manifolds

Let $M = \bigcup_p U_p$ with coordinates $x^\alpha_p$, and $N = \bigcup_q V_q$ with coordinates $y^\beta_q$ be two manifolds of dim $m$ and $n$.

**Definition 1.2.1.** A mapping $f : M \mapsto N$ is said to be smooth of smoothness class $k$ if for all $p, q$ for which $f$ determines functions $y^\beta_q(x^1_p, \ldots, x^m_p) = f(x^1_p, \ldots, x^m_p)_q^\beta$, these functions are, where defined, smooth of smoothness class $k$ (i.e. all their partial derivatives up to those of $k$th order exist and are continuous).

The smoothness class of $f$ cannot exceed the maximum class of the manifolds.

If $N = \mathbb{R}$ then $f$ is a real-valued function of the points of $M$.

A smooth mapping may not be defined on the whole manifold $M$.

E.g. each local coordinate $x^\alpha_p$ for fixed $p$ and $\alpha$ is a real-valued function on $M$ defined only on the region $U_p$.

**Definition 1.2.2.** The manifolds $M$ and $N$ are said to be smoothly equivalent or diffeomorphic if there is a one-to-one and onto map $f$ such that both $f : M \mapsto N$ and $f^{-1} : N \mapsto M$ are smooth of some class $k \geq 1$.

Since $f^{-1}$ exists then $J_{pq} = \det(\partial y^\beta_q / \partial x^\alpha_p) \neq 0$ wherever it is defined.

We always assume that the smoothness class of any manifolds and mappings are sufficiently high for our aims.
Let \( x = x(\tau), \ a \leq \tau \leq b, \) be a curve segment on a manifold \( M \ni x(\tau). \)

In \( U_p \) with \( x^\alpha_p \) it is described by the parametric equations

\[
x^\alpha_p = x^\alpha_p(\tau), \quad \alpha = 1, \ldots, m,
\]

and in \( U_p \) its velocity vector (which is tangent to the curve) is

\[
\dot{x} = (\dot{x}^1_p, \ldots, \dot{x}^m_p).
\]

In \( U_p \cap U_q \) we have two representations \( x^\alpha_p(\tau) \) and \( x^\beta_q(\tau) \) of the curve and

\[
x^\alpha_p(x^1_q(\tau), \ldots, x^m_q(\tau)) = x^\alpha_p(\tau).
\]

Thus, velocities in the two systems are related as

\[
\dot{x}^\alpha_p = \sum_{\beta=1}^{m} \frac{\partial x^\alpha_p}{\partial x^\beta_q} \dot{x}^\beta_q \equiv \frac{\partial x^\alpha_p}{\partial x^\beta_q} \dot{x}^\beta_q \quad \forall \alpha,
\]

where we use Einstein’s summation rule:

- each index can appear at most twice in any term;
- repeated indices are implicitly summed over;
- one repeated index must be upper and the other one must be lower.
Definition 1.2.3. A tangent vector to an $m$-dim manifold $M$ at an arbitrary point $x$ is represented in terms of local coordinates $x^\alpha_p$ by an $m$-tuple $(\xi^\alpha)$ of components which are linked to the components in terms of any other system $x_\beta^q$ of local coordinates as

$$
\xi^\alpha_p = \left( \frac{\partial x^\alpha_p}{\partial x_\beta^q} \right)_x \xi_\beta^q \quad \forall \alpha,
$$

(3.2)

The set of all tangent vectors to an $m$-dim manifold $M$ at a point $x$ forms an $m$-dim vector space $T_x = T_x M$, the tangent space to $M$ at the point $x$.

Thus, the velocity vector at $x$ of any smooth curve on $M$ through $x$ is a tangent vector to $M$ at $x$.

From (3.2) one sees that for any choice of local coordinates $x^\alpha$ in a neighbourhood of $x$, the operators $\frac{\partial}{\partial x^\alpha}$, operating on real-valued functions on $M$, may be thought of as forming a basis $e_\alpha = \frac{\partial}{\partial x^\alpha}$ for the tangent space $T_x$

$$
(3.2) \quad \Rightarrow \quad \xi^\alpha_p \frac{\partial}{\partial x^\alpha_p} = \xi_\beta^q \frac{\partial}{\partial x^\beta_q}.
$$
**Definition.** A smooth map \( f \) from \( M \) to \( N \) gives rise for each \( x \) to a \textit{push-forward} or \textit{an induced linear} map of tangent spaces

\[
f_* : T_x M \to T_{f(x)} N,
\]
defined as sending the velocity vector at \( x \) of any smooth curve \( x = x(\tau) \) on \( M \) to the velocity vector at \( f(x) \) to the curve \( f(x(\tau)) \) on \( N \).

In terms of local coordinates \( x^\alpha \) in a neighbourhood of \( x \in M \), and \( y^\beta \) in a neighbourhood of \( f(x) \in N \) the map \( f \) is written as

\[
y^\beta = f^\beta(x^1, \ldots, x^m), \quad \beta = 1, \ldots, n,
\]
and the push-forward map \( f_* \) as

\[
\xi^\alpha \mapsto \eta^\beta = \frac{\partial f^\beta}{\partial x^\alpha} \xi^\alpha
\]

For a real-valued function \( f : M \to \mathbb{R} \), the push-forward map \( f_* \) corresponding to each \( x \in M \) is a real-valued linear function on the tangent space to \( M \) at \( x \)

\[
\xi^\alpha \mapsto \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha,
\]
and it is represented by the gradient of \( f \) at \( x \), and is a co-vector or one-form. Thus, \( f_* \) can be identified with the differential \( df \). In particular

\[
dx^\alpha_p : \xi^\alpha \mapsto \eta = \xi^\alpha_p
\]
**Definition 1.2.4.** A *Riemann metric* on a manifold $M$ is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local coordinates of the points. Thus, at each point $x = (x_1^p, \ldots, x_m^p)$ of each chart $U_p$, the metric is given by a symmetric matrix $(g_{\alpha\beta}^{(p)}(x_1^p, \ldots, x_m^p))$, and determines a symmetric scalar product of pairs of tangent vectors at the point $x$

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)} \xi_\alpha^p \eta_\beta^p = \langle \eta, \xi \rangle, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be coordinate-independent

$$g_{\alpha\beta}^{(p)} \xi_\alpha^p \eta_\beta^p = g_{\alpha\beta}^{(q)} \xi_\alpha^q \eta_\beta^q$$

and therefore the coefficients $g_{\alpha\beta}^{(p)}$ of the quadratic form transform as

$$g_{\gamma\delta}^{(q)} = \frac{\partial x_\alpha^p}{\partial x_\gamma^q} \frac{\partial x_\beta^p}{\partial x_\delta^q} g_{\alpha\beta}^{(p)}$$

For a *pseudo-Riemann* metric on $M$ one just requires the quadratic form to be nondegenerate.
Definition 1.2.4. A tensor of type \((k, l)\) and rank \(k + l\) on an \(m\)-dim manifold \(M\) is given in each local coordinate system \(x^\alpha_p\) by a family of functions

\[
(p)T_{j_1\cdots j_l}^{i_1\cdots i_k}(x)
\]

of the point \(x\).

In other local coordinates \(x_\beta^q\) the components \((q)T_{t_1\cdots t_l}^{s_1\cdots s_k}(x)\) of the same tensor are

\[
(q)T_{t_1\cdots t_l}^{s_1\cdots s_k} = \frac{\partial x_{q_1}^{s_1}}{\partial x_{p_1}^{i_1}} \cdots \frac{\partial x_{q_k}^{s_k}}{\partial x_{p_k}^{i_k}} \cdot \frac{\partial x_{p_1}^{j_1}}{\partial x_{q_1}^{i_1}} \cdots \frac{\partial x_{p_l}^{j_l}}{\partial x_{q_l}^{i_l}} \cdot (p)T_{j_1\cdots j_l}^{i_1\cdots i_k} \tag{3.3}
\]

Let’s rewrite (3.3) as

\[
(q)T_{t_1\cdots t_l}^{s_1\cdots s_k} dx_{q_1}^{t_1} \cdots dx_{q_l}^{t_l} \frac{\partial}{\partial x_{q_1}^{s_1}} \cdots \frac{\partial}{\partial x_{q_k}^{s_k}} = (p)T_{j_1\cdots j_l}^{i_1\cdots i_k} dx_{p_1}^{j_1} \cdots dx_{p_l}^{j_l} \frac{\partial}{\partial x_{p_1}^{i_1}} \cdots \frac{\partial}{\partial x_{p_k}^{i_k}}
\]
4 Algebraic Operations on Tensors (vol 1, section 17)

1. **Permutation of indices.** Let $\sigma$ be some permutation of $1, 2, \ldots, l$

\[ \sigma = \begin{pmatrix} 1 & \cdots & l \\ \sigma(1) & \cdots & \sigma(l) \end{pmatrix} \]

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

\[ \sigma(j_1, \ldots, j_l) = (j_{\sigma(1)}, \ldots, j_{\sigma(l)}) \]

We say that a tensor $\tilde{T}_{i_1 \cdots i_k}^{j_1 \cdots j_l}$ is obtained from a tensor $T_{i_1 \cdots i_k}^{j_1 \cdots j_l}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

\[ \tilde{T}_{i_1 \cdots i_k}^{j_1 \cdots j_l} = T_{\sigma(j_1 \cdots j_l)}^{i_1 \cdots i_k} \]

Permutations of the upper indices are defined similarly.

**Example.** $\tilde{T}_{ij} = T_{\sigma(ij)} = T_{ji}$ which is a matrix transposition.
1. **Permutation of indices.** Let $\sigma$ be some permutation of 1, 2, $\ldots$, $l$

$$
\sigma = \left( \begin{array}{ccc}
1 & \cdots & l \\
\sigma(1) & \cdots & \sigma(l)
\end{array} \right)
$$

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

$$
\sigma(j_1, \ldots, j_l) = (j_{\sigma(1)}, \ldots, j_{\sigma(l)})
$$

We say that a tensor $\tilde{T}^i_1\cdots i_k$ is obtained from a tensor $T^{i_1\cdots i_k}_{j_1\cdots j_l}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$
\tilde{T}^{i_1\cdots i_k}_{j_1\cdots j_l} = T^{i_1\cdots i_k}_{\sigma(j_1\cdots j_l)}
$$

Permutations of the upper indices are defined similarly.

**Example.** $\tilde{T}^{ij} = T^{\sigma(ij)}_{\sigma(ij)} = T^{ji}$ which is a matrix transposition.

2. **Contraction (taking “traces”).** By the contraction of a tensor $T^{i_1\cdots i_k}_{j_1\cdots j_l}$ of type $(k, l)$ with respect to the indices $i_a, j_b$ we mean the tensor (summation over $n$)

$$
\tilde{T}^{i_1\cdots i_{k-1}}_{j_1\cdots j_{l-1}} = T^{i_1\cdots i_{a-1}i_{a+1}\cdots i_k}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_l}
$$

of type $(k - 1, l - 1)$.

**Example.** $T^n = \text{tr} T$ of the matrix $T^i_j$. 

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22
1. **Permutation of indices.** Let \( \sigma \) be some permutation of \( 1, 2, \ldots, l \)

\[
\sigma = \begin{pmatrix}
1 & \cdots & l \\
\sigma(1) & \cdots & \sigma(l)
\end{pmatrix}
\]

\( \sigma \) acts on the ordered \( l \)-tuple \((j_1, \ldots, j_l)\) as

\[
\sigma(j_1, \ldots, j_l) = (j_{\sigma(1)}, \ldots, j_{\sigma(l)})
\]

We say that a tensor \( \tilde{T}^{i_1 \cdots i_k}_{j_1 \cdots j_l} \) is obtained from a tensor \( T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \) by means of a permutation \( \sigma \) of the lower indices if at each point of \( M \)

\[
\tilde{T}^{i_1 \cdots i_k}_{j_1 \cdots j_l} = T^{i_1 \cdots i_k}_{\sigma(j_1 \cdots j_l)}
\]

Permutations of the upper indices are defined similarly.

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\tilde{T}^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{l-1}} = T^{i_1 \cdots i_{a-1}n_{i_a+1} \cdots i_k}_{j_1 \cdots j_{b-1}n_{j_b+1} \cdots j_l}
\]

of type \((k - 1, l - 1)\).

**Example.** \( T^n = \text{tr}\, T \) of the matrix \( T^i_j \).

3. **Product of tensors.** Given two tensors \( T = (T^{i_1 \cdots i_p}_{j_1 \cdots j_q}) \) of type \((p, q)\) and \( P = (P^{i_1 \cdots i_k}_{j_1 \cdots j_l}) \) of type \((k, l)\), we define their product to be the tensor \( S = T \otimes P \) of type \((p + k, q + l)\) with components

\[
S^{i_1 \cdots i_{p+k}}_{j_1 \cdots j_{q+l}} = T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \cdot P^{i_{p+1} \cdots i_{p+k}}_{j_{q+1} \cdots j_{q+l}}
\]

This multiplication is not commutative but it is associative.
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^{ij} = \xi^i \eta_j$ of type $(1,1)$. Contraction $T^i_i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_l \Rightarrow T^{ik} = \xi^i A^k_l$ of type $(2,1)$. Contraction $\eta^k = \xi^i A^k_i$ is a vector, the result of applying the linear transformation $A^k_l$ to the vector.
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^i_j = \xi^i \eta_j$ of type $(1,1)$. Contraction $T_i^i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_i \Rightarrow T^{ik}_i = \xi^i A^k_i$ of type $(2,1)$. Contraction $\eta^k = \xi^i A^k_i$ is a vector, the result of applying the linear transformation $A^k_i$ to the vector.

Example 3. We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows:

Since the gradient $\frac{\partial f}{\partial x^i}$ of a function $f$ is a co-vector, the quantity

$$\partial_{\xi} f = \xi^i \frac{\partial f}{\partial x^i}$$

is a scalar called the directional derivative of $f$ in the direction of $\xi$.

Thus, an arbitrary vector $\xi$ corresponds to the operator

$$\partial_{\xi} = \xi^i \frac{\partial}{\partial x^i}$$

We identify $e_i = \frac{\partial}{\partial x^i}$ with the canonical basis of the tangent space.
5 Tensors of type \((0, k)\) (vol 1, section 18)

These are tensors with lower indices: \(T_{i_1 \cdots i_k}\)

5.1 Co-vectors: Tensors of type \((0, 1)\)

The gradient \(\left(\frac{\partial f}{\partial x^i}\right)\) of a function \(f\) is the standard example.

Recall that the differential of a function \(f\) of \(x^1, \ldots, x^n\) corresponding to increments \(dx^i\) in the \(x^i\) is

\[
\text{df} = \frac{\partial f}{\partial x^i} dx^i
\]

Since \(dx^i\) is a vector \(df\) has the same value in any coordinate system.

In general, given any co-vector \((T_i)\), the differential form \(T_i dx^i\) is invariant under a change of a chart.

We identify \(dx^i \equiv e^i\) with the canonical basis of co-vectors or cotangent space.
5.2 Tensors of type $(0, 2)$

A basis for the space of tensors of type $(0, 2)$ at a given point are the products

$$e^i \otimes e^j$$

In terms of this basis an arbitrary tensor $T_{ij}$ has the form

$$T_{ij} e^i \otimes e^j$$

and can be regarded as a bilinear form on vectors since if $\xi, \eta$ are vectors then the scalar

$$T_{ij} \xi^i \eta^j$$

can be considered as the value of the bilinear form on those vectors.
Any $T_{ij}$ can be expressed as

\[ T_{ij} = T^{\text{sym}}_{ij} + T^{\text{alt}}_{ij} \]

\[ T^{\text{sym}}_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) = T^{\text{sym}}_{ji} \]

\[ T^{\text{alt}}_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) = -T^{\text{alt}}_{ji} \]

A basis of $T^{\text{sym}}_{ij}$ is $\frac{e^i \otimes e^j + e^j \otimes e^i}{2}$, $i \leq j$

A basis of $T^{\text{alt}}_{ij}$ is $e^i \otimes e^j - e^j \otimes e^i$, $i < j$

Then

\[ T^{\text{sym}}_{ij} e^i \otimes e^j = T^{\text{sym}}_{ij} \frac{e^i \otimes e^j + e^j \otimes e^i}{2} \]

\[ = \sum_i T^{\text{sym}}_{ii} e^i \otimes e^i + \sum_{i<j} 2T^{\text{sym}}_{ij} \frac{e^i \otimes e^j + e^j \otimes e^i}{2} \]

and

\[ T^{\text{alt}}_{ij} e^i \otimes e^j = T^{\text{alt}}_{ij} \frac{e^i \otimes e^j - e^j \otimes e^i}{2} \]

\[ = \sum_{i<j} T^{\text{alt}}_{ij} (e^i \otimes e^j - e^j \otimes e^i) \]

In differential notation we identify

\[ \frac{e^i \otimes e^j + e^j \otimes e^i}{2} \equiv dx^i dx^j = dx^j dx^i \]

\[ e^i \otimes e^j - e^j \otimes e^i \equiv dx^i \wedge dx^j = -dx^j \wedge dx^i \]
5.3 Skew-symmetric Tensors of type $(0, k)$

**Definition.** A *skew-symmetric* tensor of type $(0, k)$ is a tensor $T_{i_1\cdots i_k}$ satisfying

$$T_{\sigma(i_1\cdots i_k)} = s(\sigma)T_{i_1\cdots i_k}$$

where for all permutations $\sigma$

$$s(\sigma) = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \end{cases}$$

Thus, if two indices are equal then the corresponding component of $T_{i_1\cdots i_k}$ is equal to 0.

Then, if $k > n$ the tensor is identically 0.

In what follows we assume $k \leq n$. 
The standard basis at a given point is

\[ dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k \]

where

\[ dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathcal{S}(\sigma) e^{\sigma(i_1)} \otimes \cdots \otimes e^{\sigma(i_k)} \]

Here \( S_k \) is the symmetric group, i.e. the group of all permutations of \( 1, \ldots, k \), and \( \sigma(i_l) \equiv i_{\sigma(l)} \).

The differential form of the skew-symmetric tensor \((T_{i_1\cdots i_k})\) is

\[ T_{i_1\cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = \sum_{i_1 < \cdots < i_k} T_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]

\[ = \frac{1}{k!} T_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]
Example 1. A skew-symmetric tensor $T_{i_1...i_n}$ of type $(0, n)$ in $n$-dim manifold is determined by the single component $T_{1...n}$

$$T_{\sigma(1...n)} = s(\sigma)T_{1...n}$$

Thus, the space of skew-symmetric tensors of type $(0, n)$ is one-dim.

We denote $T_{i_1...i_n}$ with $T_{1...n} = 1$ as

$$\epsilon_{i_1...i_n}$$

It is called the \textit{Levi-Civita} symbol (or tensor) of rank $n$.

\textbf{Theorem.} Skew-symmetric tensors of type $(0, n)$ where $n$ is the dimension of the manifold $M$ transform as

$$(p)^{T_{1...n}} = (q)T_{1...n} \cdot J$$

where $J$ is the Jacobian

$$J = \det \left( \frac{\partial x^i_q}{\partial x^j_p} \right)$$

HW: Prove the theorem
Example 2.
Let $G = (g_{ij})$ be a non-degenerate tensor, i.e. $g \equiv \det(g_{ij}) \neq 0$ (it does not have to be symmetric).

Then
\[
g_{ij}^{(p)} = \frac{\partial x^k_q}{\partial x^i_p} \frac{\partial x^l_q}{\partial x^j_p} g_{kl}^{(q)} = \frac{\partial x^k_q}{\partial x^i_p} g_{kl}^{(q)} \frac{\partial x^l_q}{\partial x^j_p}
\]

or in matrix notations
\[
g_{ij}^{(p)} = (A^T G^{(q)} A)_{ij}, \quad A = (A^l_i) = \left(\frac{\partial x^l_q}{\partial x^i_p}\right) \implies g^{(p)} = (\det A)^2 g^{(q)}
\]

Thus, if $\det A > 0$ then
\[
\sqrt{|g^{(p)}|} = \sqrt{|g^{(q)}|} \det A = \sqrt{|g^{(q)}|} J
\]

Comparing with
\[
T^{(p)}_{1...n} = T^{(q)}_{1...n} \cdot J
\]

one concludes

The expression $\sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^n$ behaves as a tensor under coordinate changes for which the Jacobian $J = \det \left(\frac{\partial x^i_q}{\partial x^p}ight)$ is positive.

Since $J > 0$, the manifold $M$ is oriented.
Metric and distance function.

A metric $g_{ij}$ on a manifold is a tensor of type $(0, 2)$, and on an oriented manifold such a metric gives rise to a volume element

$$T_{i_1\cdots i_k} = \sqrt{|g|} \epsilon_{i_1\cdots i_k}, \quad g = \det(g_{ij})$$

It is convenient to write the volume element in the notation of differential forms

$$\Omega = \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^n$$

If $g_{ij}$ is Riemann then the volume $V$ of $M$ is

$$V = \int_M \Omega = \int_M \sqrt{g} \, dx^1 \wedge \cdots \wedge dx^n$$

A Riemann metric $ds^2 = g_{ij} dx^i dx^j$ on a connected manifold $M$ gives rise to a metric space structure on $M$ with distance function $\rho(x, y)$ defined by

$$\rho(x, y) = \inf_{\gamma} \int_{\gamma} ds$$

where the infimum is taken over all piece-wise smooth arcs joining the points $x$ and $y$.

The topology on $M$ defined by this metric-space structure coincides with the Euclidean topology on $M$. 
6 Embeddings and Immersions of Manifolds

**Definition 1.3.1a.** A manifold $M$ of dim $m$ is said to be immersed in a manifold $N$ of dim $n \geq m$ if $\exists$ a smooth map $f : M \hookrightarrow N$ such that the push-forward map $f_*$ is at each point a one-to-one map of the tangent space. The map $f$ is called an immersion of $M$ into $N$.

Since $f_*$ is at each point a one-to-one map of the tangent space, in terms of local coordinates the Jacobian matrix of $f$ at each point has rank equal to $m = \dim M$.

**Definition 1.3.1b.** An immersion of $M$ into $N$ is called embedding if it is one-to-one. Then, $M$ is called a submanifold of $N$. 
We always assume that any submanifold $M$ is defined in each chart $U_p$ of the containing manifold $N$ by a system of eqs

$$\begin{align*}
&f^1_p(x^1_p, \ldots, x^n_p) = 0 \\
f^2_p(x^1_p, \ldots, x^n_p) = 0 \\
&\vdots \\
f^{n-m}_p(x^1_p, \ldots, x^n_p) = 0
\end{align*}$$

where $\text{rank} \left( \frac{\partial f^i_p}{\partial x^\alpha_p} \right) = n - m$

with the property that on each intersection $U_p \cap U_q$ the systems $(f^i_p = 0)$ and $(f^i_q = 0)$ have the same set of zeroes.

Let us introduce in each $U_p \subset N$ new coordinates $y^1_p, \ldots, y^n_p$ satisfying

$$y^{m+1}_p = f^1_p(x^1_p, \ldots, x^n_p), \quad y^{m+2}_p = f^2_p(x^1_p, \ldots, x^n_p), \ldots, \quad y^n_p = f^{n-m}_p(x^1_p, \ldots, x^n_p)$$

Then, $M$ is given by

$$y^{m+1}_p = 0, \quad y^{m+2}_p = 0, \quad \cdots, \quad y^n_p = 0$$

while $y^1_p, \ldots, y^m_p$ serve as local coordinates on $M$. 

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Definition 1.3.2. A closed region $A$ of a manifold $M$ defined by an inequality $f(x) \leq 0$ (or $f(x) \geq 0$) where $f$ is a real-valued function on $M$ is called a manifold with boundary.

It is assumed that the boundary $\partial A$ given by $f(x) = 0$ is a non-singular submanifold of $M$, i.e. $\vec{\nabla} f \neq 0$ on $\partial A$.

Let $A$ and $B$ be manifolds with boundary, both given as closed regions of manifolds $M$ and $N$. A map

$$\varphi : A \mapsto B$$

is said to be a smooth map of manifolds with boundary if it is a restriction to $A$ of a smooth map

$$\tilde{\varphi} : U \mapsto N, \quad \tilde{\varphi}|A = \varphi$$

of an open region $U$ of $M$ containing $A$, e.g. if

$$A : f(x) \leq 0 \text{ then } U \text{ is } U_\epsilon = \{x|f(x) < \epsilon\}, \quad \epsilon > 0.$$  

Definition. A compact manifold without boundary is called closed.
7 Surfaces in Euclidean space

7.1 Surfaces as Manifolds

Definition. A non-singular surface $M$ of dimension $k$ in $n$-dim Euclidean space is given by a set of $n - k$ eqs

$$f_i(x^1, \ldots, x^n) = 0, \quad i = 1, \ldots, n - k$$ (7.4)

where $\forall x$ the matrix $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ has rank $n - k$.

Let $J_{j_1 \cdots j_{n-k}}$ be the minor of a submatrix made up of the columns of $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ which are indexed by $j_1, \ldots, j_{n-k}$.

Let $U_{j_1 \cdots j_{n-k}}$ be the region consisting of all points of the surface at which $J_{j_1 \cdots j_{n-k}}$ does not vanish.

Obviously,

$$M = \bigcup_{j_1 \cdots j_{n-k}} U_{j_1 \cdots j_{n-k}}$$

Since $J_{j_1 \cdots j_{n-k}} \neq 0$ on $U_{j_1 \cdots j_{n-k}}$, we can take

$$(y^1, \ldots, y^k) = (x^1, \ldots, \hat{x}^{j_1}, \ldots, \hat{x}^{j_{n-k}}, \ldots, x^n)$$ (7.5)

as local coordinates on $U_{j_1 \cdots j_{n-k}}$.

**Theorem 2.1.1.** The covering of the surface $M$ (7.4) by the regions

$$U_{j_1 \cdots j_{n-k}}, \quad 1 \leq j_1 < \cdots < j_{n-k} \leq n$$

each furnished with local coordinates (7.5), defines on the surface the structure of a smooth manifold.

**Proof.** Blackboard
**Remark 1.** The Jacobian of the transition function \( y \to z \) is

\[
J_{(y)\to(z)} = \pm \frac{J_{s_1 \cdots s_{n-k}}}{J_{j_1 \cdots j_{n-k}}}
\]

HW: Prove it.

**Remark 2.** The tangent space to the surface \( M \) \( (7.4) \) is identifiable with the linear subspace of \( \mathbb{R}^n \) consisting of the solutions of the system

\[
\frac{\partial f_1}{\partial x^\alpha} \xi^\alpha = 0, \ldots, \frac{\partial f_{n-k}}{\partial x^\alpha} \xi^\alpha = 0
\]

Thus, the co-vectors \( \vec{\nabla} f_i = \left( \frac{\partial f_i}{\partial x^\alpha} \right), i = 1, \ldots, n - k \) are orthogonal to the surface at each point.
7.2 Surfaces can be oriented

Consider at any point \( x \) of an \( n \)-dim manifold \( M \) the various frames (i.e. ordered bases)

\[
\tau = (e_1, \ldots, e_n)
\]

for the tangent space to \( M \) at \( x \). Any two such frames

\[
\tau_1 = (e_1^{(1)}, \ldots, e_n^{(1)}) \quad \text{and} \quad \tau_2 = (e_1^{(2)}, \ldots, e_n^{(2)})
\]

are related by a nonsingular linear transformation \( A \)

\[
A : e_k^{(1)} \rightarrow e_k^{(2)}, \quad k = 1, \ldots, n
\]

**Definition.** We say the ordered bases \( \tau_1, \tau_2 \) lie in the same orientation class if \( \det A > 0 \), and lie in opposite orientation classes if \( \det A < 0 \).

**Definition 2.1.2.** A manifold is said to be **orientable** if it is possible to choose at every point of it a single orientation class depending continuously on the points.

A particular choice of such an orientation class for each point is called an **orientation** of the manifold, and a manifold equipped with a particular orientation is said to be **oriented**.

If no orientation exists the manifold is **non-orientable**.
**Theorem 2.1.3.** Definition 1.1.3 is equivalent to Definition 2.1.2.

**Proof of Def 1.1.3 ⇒ Def 2.1.2.**

Let $M$ be oriented in the sense of Def 1.1.3. Then, we choose at each $x \in U_j \subset M$ as our orienting frame the $n$-tuple

$$(e_{1j}, \ldots, e_{nj})$$

consisting of the standard basis vectors tangent to the coordinate axes of the local coordinate system

$$x_j^1, \ldots x_j^n$$

If

$$x \in U_j \quad \text{and} \quad x \in U_k$$

then the two orienting frames are related by the Jacobian matrix of transition function. Since the Jacobian is positive the two frames lie in the same orientation class.

**Proof of Def 1.1.3 ⇒ Def 2.1.2.** See the textbook
**Theorem 2.1.4.** A smooth non-singular surface $M^k$ in $n$-dim space $\mathbb{R}^n$, defined by a system of eqs (7.4), is orientable.

**Proof.** Let $\tau$ denote a point-dependent tangent frame to the surface $M^k$. The ordered $n$-tuple

$$\hat{\tau} = (\tau, \nabla f_1, \ldots, \nabla f_{n-k})$$

of vectors is linearly independent at each point because $\nabla f_i$ are linearly independent among themselves and orthogonal to the surface.

We can choose $\tau$ at each $x \in M^k$ so that $\hat{\tau}$ lies in the same orientation class as the standard frame

$$(e_1, \ldots, e_n)$$

Since this orientation class depends continuously on $x \in \mathbb{R}^n$, so will the orientation class of $\tau$ depend continuously on $x \in M^k$. 
Example 1. An $n$-sphere $S^n$ in $\mathbb{R}^{n+1}$: $x_1^2 + \cdots + x_{n+1}^2 = 1$.

The $n$-sphere bounds a manifold with boundary, denoted by $D^{n+1}$ and called the closed $(n+1)$-dim disc or ball, defined by

$$f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1 \leq 0$$

Then $S^n$ separates $\mathbb{R}^{n+1}$ into two non-intersecting regions defined by $f(x) < 0$ and $f(x) > 0$.

Example 2. A hyperbolic $n$-space, $H^n$, is one sheet of the hyperboloid of two sheets realised as a surface

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = -1, \quad x_0 > 0,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 3. An $n$-dimensional de Sitter space, $dS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = 1,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 4. An $n$-dimensional anti-de Sitter space, $AdS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_{-1}^2 - x_0^2 + \sum_{i=1}^{n-1} x_i^2 = -1,$$

in the pseudo-Euclidean space $\mathbb{R}^{2,n-1}$
Definition 2.1.5. A connected \((n - 1)\)-dim submanifold of \(\mathbb{R}^n\) is called \textit{two-sided} if a single-valued continuous field of unit normals can be defined on it. Such a submanifold is called a two-sided hypersurface.

Theorem 2.1.6. A two-sided hypersurface in \(\mathbb{R}^n\) is orientable.

Proof. See the textbook

It can be shown that any two-sided hypersurface in \(\mathbb{R}^n\) is defined by a single non-singular eq \(f(x) = 0\). Thus, it bounds a manifold with boundary.

Then, one can prove that any closed hypersurface in \(\mathbb{R}^n\) is two-sided.
7.3 Transformation Groups as Surfaces

**Definition.** A *group* is a nonempty set $G$ on which there is defined a binary operation $(a, b) \mapsto ab$ satisfying the following properties

- **Closure:** If $a$ and $b$ belong to $G$, then $ab$ is also in $G$.
- **Associativity:** $a(bc) = (ab)c$ for all $a, b, c \in G$.
- **Identity:** There is an element $1 \in G$ such that $a1 = 1a = a$ for all $a$ in $G$.
- **Inverse:** If $a \in G$, then there is an element $a^{-1} \in G$: $aa^{-1} = a^{-1}a = 1$.

Examples of groups which are manifolds are

1. The *general* linear group $GL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with non-zero determinant is a region in $\mathbb{R}^{n^2}$.

2. The *special* linear group $SL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with determinant equal to 1

   $$\det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a hypersurface in $\mathbb{R}^{n^2}$.

3. The *orthogonal* group $O(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying

   $$A^T \cdot A = I, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a surface in $\mathbb{R}^{n^2}$.

4. The *special orthogonal* group $SO(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying

   $$A^T \cdot A = I, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a surface in $\mathbb{R}^{n^2}$.
5. The \textit{pseudo-orthogonal} group $O(p, q, \mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices satisfying

$$A^T \cdot \eta \cdot A = \eta, \quad \eta = \text{diag}\left(1, \ldots, 1, -1, \ldots, -1\right), \quad A \in \text{Mat}(n, \mathbb{R})$$

is a surface in $\mathbb{R}^{n^2}$.

6. The \textit{special pseudo-orthogonal} group $SO(p, q, \mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices $A \in \text{Mat}(n, \mathbb{R})$ satisfying

$$A^T \cdot \eta \cdot A = \eta, \quad \det A = 1, \quad \eta = \text{diag}\left(1, \ldots, 1, -1, \ldots, -1\right)$$

is a surface in $\mathbb{R}^{n^2}$.

7. The \textit{unitary} group $U(n)$ consisting of all $n \times n$ complex matrices satisfying

$$A^\dagger \cdot A = \mathbb{I}, \quad A \in \text{Mat}(n, \mathbb{C})$$

is a surface in $\mathbb{R}^{2n^2}$.

8. E.g. $U(1) \simeq S^1$.

9. The \textit{special unitary} group $SU(n)$ consisting of all $n \times n$ complex matrices satisfying

$$A^\dagger \cdot A = \mathbb{I}, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{C})$$

is a surface in $\mathbb{R}^{2n^2}$.

10. E.g. $SU(2) \simeq S^3$.

11. The \textit{pseudo-unitary} group $U(p, q)$ consisting of all $n \times n$, $n = p+q$ complex matrices satisfying

$$A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag}\left(1, \ldots, 1, -1, \ldots, -1\right), \quad A \in \text{Mat}(n, \mathbb{C})$$

is a surface in $\mathbb{R}^{2n^2}$. 

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12. The *special pseudo-unitary* group $SU(p, q)$ consisting of all $n \times n$, 
$n = p + q$ complex matrices $A \in \text{Mat}(n, \mathbb{C})$ satisfying

$$A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag}(1, \ldots, 1, -1, \ldots, -1), \quad \det A = 1$$

is a surface in $\mathbb{R}^{2n^2}$

**Definition 2.1.7.** A manifold $G$ is called a *Lie group* if it has given on it a group operation with the property that the maps $\varphi : G \mapsto G$, defined by $\varphi(g) = g^{-1}$ (i.e. the taking of inverses) and $\psi : G \times G \mapsto G$ defined by $\psi(g, h) = gh$ (i.e. the group multiplication), are smooth maps.
8 Projective Spaces

8.1 Real Projective Space

**Definition.** The *real projective space* $\mathbb{R}P^n$ is the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin. Equivalently it is the set of equivalence classes of nonzero vectors in $\mathbb{R}^{n+1}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.

Since each line passing through the origin intersect a sphere $S^n$ centred at the origin in exactly two points, the points of $\mathbb{R}P^n$ are in one-to-one correspondence with the pairs of diametrically opposite points of the $n$-sphere. We may think of $\mathbb{R}P^n$ as obtained from $S^n$ by gluing, that is identifying, diametrically opposite points. So, $\mathbb{R}P^n \cong S^n/Z_2$, where $Z_2$ maps a point of $S^n$ to the diametrically opposite point.

The projective line

$$\mathbb{R}P^1 \cong S^1/Z_2 \cong S^1 \cong U(1)$$

$\mathbb{R}P^2$ is called the projective plane
8.2 Quaternions, $SU(2)$, $SO(3)$, $\mathbb{R}P^3$

**Definition.** The set $\mathbb{H}$ of *quaternions* consists of all linear combinations

$$q \in \mathbb{H}, \quad q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R},$$

and $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are linearly independent (so it is a 4-dim vector space).

We introduce the following multiplication in $\mathbb{H}$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{i} = -\mathbf{k} \cdot \mathbf{j}, \quad \mathbf{k} \cdot \mathbf{i} = \mathbf{j} = -\mathbf{i} \cdot \mathbf{k},$$

$$\mathbf{i} \cdot \mathbf{i} \equiv \mathbf{i}^2 = -\mathbf{1}, \quad \mathbf{j} \cdot \mathbf{j} \equiv \mathbf{j}^2 = -\mathbf{1}, \quad \mathbf{k} \cdot \mathbf{k} \equiv \mathbf{k}^2 = -\mathbf{1},$$

$$\mathbf{i} \cdot \mathbf{1} = \mathbf{i} = \mathbf{1} \cdot \mathbf{i}, \quad \mathbf{j} \cdot \mathbf{1} = \mathbf{j} = \mathbf{1} \cdot \mathbf{j}, \quad \mathbf{k} \cdot \mathbf{1} = \mathbf{k} = \mathbf{1} \cdot \mathbf{k}, \quad \mathbf{1} \cdot \mathbf{1} = \mathbf{1},$$

which makes $\mathbb{H}$ an associative algebra over the field of real numbers.
For each quaternion

\[ q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R}, \]

we define

\[ A(q) = \begin{pmatrix} a + b i & c + d i \\ -c + d i & a - b i \end{pmatrix}, \quad A(q) \in \text{Mat}(2, \mathbb{C}). \]

**Lemma 1.** The map \( q \mapsto A(q) \) is one-to-one and

\[ A(q_1 q_2) = A(q_1)A(q_2) \]

so that this map is an algebra *monomorphism* which means it is an injective (one-to-one) homomorphism (consistent with multiplication).

Note

\[ A(\mathbf{i}) = i\sigma_3, \quad A(\mathbf{j}) = i\sigma_2, \quad A(\mathbf{k}) = i\sigma_1, \]

where \( \sigma_i \) are the Pauli matrices.
We define an operation of *conjugation* on $\mathbb{H}$ by

$$\bar{q} = a \mathbf{1} - b \mathbf{i} - c \mathbf{j} - d \mathbf{k}, \quad a, b, c, d \in \mathbb{R}$$

**Lemma 2.** The map $q \mapsto \bar{q}$ is an anti-isomorphism of $\mathbb{H}$, i.e. it is linear, and

$$\bar{q_1 q_2} = \bar{q_2} \bar{q_1}$$

We define the *norm* $|q| \geq 0$ of a quaternion by

$$|q|^2 = q \bar{q} = a^2 + b^2 + c^2 + d^2, \quad a, b, c, d \in \mathbb{R}$$

Then, if $|q| \neq 0$ then

$$q \cdot \frac{\bar{q}}{|q|^2} = \mathbf{1}$$

and therefore each nonzero quaternion has multiplicative inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

**Lemma 3.** The algebra $\mathbb{H}$ of quaternions is a *division* algebra, i.e. for each nonzero quaternion $q$ there exists a quaternion $q^{-1}$ inverse to it

$$qq^{-1} = \mathbf{1} = q^{-1}q$$
We have
\[ |q|^2 = \det A(q) \implies |q_1q_2|^2 = |q_1|^2|q_2|^2 \]
Thus, the set of quaternions with norm 1 forms a group under multiplication. It is denoted by \( \mathbb{H}_1 \).

For \( q \in \mathbb{H}_1 \) we have \( q^{-1} = \bar{q} \).

If we regard \( \mathbb{H} \) as \( \mathbb{R}^4 \) then \( \mathbb{H}_1 \) is the hypersurface
\[ |q|^2 = a^2 + b^2 + c^2 + d^2 = 1 \implies \mathbb{H}_1 \cong S^3 \cong SU(2) \]
Let $\mathbb{H}_0$ denote the 3-dim Euclidean space consisting of all quaternions $x$ satisfying $\bar{x} = -x$, i.e. with the real part zero. Then

$$|x|^2 = x \cdot \bar{x} = -x^2 \geq 0$$

**Lemma 4.** If $|q|^2 = 1$, then the transformation defined by

$$\alpha_q : x \mapsto qxq^{-1}, \quad x \in \mathbb{H}_0$$

is a rotation of 3-dim Euclidean space $\mathbb{H}_0 = \mathbb{R}^3$.

Thus, the map $q \mapsto \alpha_q$ is a homomorphism from $\mathbb{H}_1 \cong SU(2)$ to the group of rotations of $\mathbb{R}^3$.

Obviously, $\alpha_q$ is the identity map when $q = \pm 1$.

Check that every rotation is of the form (8.6).

Thus, $SO(3)$ is isomorphic to $SU(2)/\{1, -1\}$.

Since $\{1, -1\} \cong Z_2$, where $Z_2$ identifies $A$ and $-A$ which correspond to diametrically opposite points of $S^3 \cong SU(2)$, we get

$$SO(3) \cong SU(2)/Z_2 \cong S^3/Z_2 \cong \mathbb{R}P^3$$

**Remark.** $\mathbb{H}_1 \cong SU(2)$ can be also used to prove that

$$SO(4) \cong (SU(2) \times SU(2))/Z_2$$
8.3 \( \mathbb{R}P^n \) as a manifold

Let's introduce a manifold structure on \( \mathbb{R}P^n \).

\( \mathbb{R}P^n \) as the set of equivalence classes of nonzero vectors in \( \mathbb{R}^{n+1} \) with coordinates \( y^0, \ldots, y^n \).

For each \( q = 0, 1, \ldots, n \), let \( U_q \) denote the set of equivalence classes of vectors \((y^0, \ldots, y^n)\) with \( y^q \neq 0 \). Obviously, \( \mathbb{R}P^n = \bigcup_{q=0}^{n} U_q \).

We introduce the following local coordinates on \( U_0 \)

\[
    x_i^0 = \frac{y_i}{y^0}, \quad i = 1, \ldots, n.
\]  

(8.7)

Then, the local coordinates on each \( U_q \) are introduced through the following recursion relations

\[
    x_{q+1}^i = \frac{x_q^i}{x_1^q}, \quad i = 1, \ldots, n - 1, \quad x_{q+1}^n = -\frac{1}{x_1^q}.
\]  

(8.8)

One gets

\[
    x_1^q = \frac{x_2^{q-1}}{x_1^{q-1}} = \frac{x_3^{q-2}}{x_2^{q-2}} = \frac{x_4^{q-3}}{x_3^{q-3}} = \cdots
\]  

\[
    = \frac{x_{q-p}^p}{x_{q-p}^p} = \frac{x_0^{q+1}}{x_0^q} = \frac{y^{q+1}}{y^q},
\]  

(8.9)

and therefore \( x_1^q \neq 0 \) on \( U_{q+1} \) and up to a sign \( x_{q+1}^i \) can be expressed as the ratio \( y^{n_i}/y^{q+1} \) where the index \( n_i \) depends on \( i \) and \( q \). Eqs (8.8) also provide transition functions on the intersection \( U_q \cap U_{q+1} \).
\[ x_{q+1}^i = \frac{x_{q+1}^{i+1}}{x_q^1}, \quad i = 1, \ldots, n - 1, \quad x_{q+1}^n = -\frac{1}{x_q^1}. \quad (8.10) \]

\[ x_q^1 = \frac{y_q^{q+1}}{y_q^{q}} \quad (8.11) \]

The Jacobian of these transition functions can be easily found

\[ J_{(x_q) \to (x_{q+1})} = \left( -\frac{1}{x_q^1} \right)^{n+1} \neq 0. \quad (8.12) \]

Since, the Jacobian of the transition functions for arbitrary \( q \) and \( p \) is

\[ J_{(x_q) \to (x_p)} = J_{(x_q) \to (x_{q+1})}J_{(x_{q+1}) \to (x_{q+2})} \cdots J_{(x_{p-1}) \to (x_p)} \]

\[ = \prod_{k=q}^{p-1} \left( -\frac{1}{x_k^1} \right)^{n+1} = \prod_{k=q}^{p-1} \left( -\frac{y_k^k}{y_{k+1}^{k+1}} \right)^{n+1} \]

\[ = \left( (-1)^{p-q} \frac{y_q^q}{y_p^p} \right)^{n+1} \neq 0, \quad (8.13) \]

and it is a smooth function on \( U_q \cap U_p \), we conclude that the real projective space \( \mathbb{R}P^n \) is a smooth manifold.

Moreover, \( \mathbb{R}P^n \) is oriented for odd \( n \).

\( \mathbb{R}P^2 \) is called the projective plane, and \( U_0 \) is called the finite part of the projective plane.
8.4 Complex Projective Space

**Definition.** The *complex projective space* \( \mathbb{C}P^n \) is the set of equivalence classes of nonzero vectors in \( \mathbb{C}^{n+1} \) where two nonzero vectors are equivalent if they are scalar multiples of one another.

The charts are defined as in the real case, making \( \mathbb{C}P^n \) a \( 2n \)-dim smooth manifold.

Consider the *complex projective line* \( \mathbb{C}P^1 \). Its points are equivalence classes of nonzero pairs

\[
(z^0, z^1) \sim (\lambda z^0, \lambda z^1), \quad \lambda \neq 0, \ \lambda \in \mathbb{C}
\]

Consider the complex function

\[
w_0(z^0, z^1) = \frac{z^1}{z^0}
\]

defined on \( U_0 : z^0 \neq 0 \) which covers all \( \mathbb{C}P^1 \) except the equivalence class of \((0, 1)\) which contains all nonzero pairs of the form \((0, z^1)\).

We define \( w_0 \) as taking the value \( \infty \) at this point.

Then, via the function \( w_0 \), \( \mathbb{C}P^1 \) becomes identified with the “extended complex plane”, i.e. the union of the ordinary complex plane with a point at infinity.

**Theorem 2.2.1.** The complex projective line \( \mathbb{C}P^1 \) is diffeomorphic to the 2-dim sphere \( S^2 \).

**Proof.** Blackboard

By this result, the extended complex plane is often called the Riemann sphere.
The general complex projective space $\mathbb{C}P^n$

From each equivalence class of $(n + 1)$-dim vectors $(z^0, z^1, \ldots, z^n)$ we choose as a representative a vector with the unit norm

$$|z^0|^2 + |z^1|^2 + \cdots |z^n|^2 = 1$$

This equation defines the unit sphere $S^{2n+1}$ in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.

This representative is not unique because one can multiply all $z^i$ by $e^{i\varphi}$, $\varphi \in \mathbb{R}$, i.e. by complex numbers of modulus 1 which form the group $U(1)$.

Thus, the complex projective space $\mathbb{C}P^n$ can be obtained from the unit sphere $S^{2n+1} = \{z| \sum_{i=0}^{n} |z^i|^2 = 1\}$, by identifying all points differing by a scalar factor of the form $e^{i\varphi}$.

This identification provides a map

$$S^{2n+1} \mapsto \mathbb{C}P^n \cong S^{2n+1}/U(1)$$

such that the pre-image of each point of $\mathbb{C}P^n$ is topologically equivalent to the circle $S^1$.

In particular, since $\mathbb{C}P^1 \cong S^2$, we get a map

$$S^3 \mapsto S^2, \quad (z^0, z^1) \mapsto w = \frac{z^1}{z^0}, \quad |z^0|^2 + |z^1|^2 = 1$$

This map is called the Hopf bundle, or fibration or map. $S^1$ is the fibre space embedded in $S^3$, and the Hopf map projects $S^3$ onto the base space $S^2$. 

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9 Elements of the Theory of Lie Groups

9.1 Neighbourhood of the Identity Element of a Lie Group

Let $G$ be a Lie group, let the point $g_0 = 1 \in G$ be the identity element of $G$, and let $T = T_{(1)}$ be the tangent space at the identity element.

Let us express the group operations on $G$ in a chart $U_0$ containing $g_0$ in terms of local coordinates.

We choose coordinates in $U_0$ so that the identity element is the origin

$$1 = g_0 = (0, \ldots, 0)$$

Let

$$g_1 = (x^1, \ldots, x^n), \quad g_2 = (y^1, \ldots, y^n), \quad g_3 = (z^1, \ldots, z^n)$$

where $g_k$ are such that

$$g_k, g_k^{-1}, g_k g_l, g_k^{-1} g_l^{-1}, g_k g_l^{-1}, g_k^{-1} g_l \in U_0, \quad k, l = 1, 2, 3$$

Then,

$$g_1 g_2 = (\psi^1(x, y), \psi^2(x, y), \ldots, \psi^n(x, y)) = (\psi^i(x, y)), \quad \psi^i(x, y) = \psi^i(x^1, \ldots, x^n, y^1, \ldots, y^n), \quad i = 1, \ldots, n$$

$$g_1^{-1} = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^n(x)) = (\varphi^i(x)), \quad \varphi^i(x) = \varphi^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n$$

are the coordinates of $g_1 g_2$ and $g_1^{-1}$.

$\psi(x, y)$ and $\varphi(x)$ satisfy ($i = 1, \ldots, n$)

1. $\psi^i(x, 0) = \psi^i(0, x) = x^i$

2. $\psi^i(x, \varphi(x)) = 0$

3. $\psi^i(x, \psi(y, z)) = \psi^i(\psi(x, y), z)$
Let \( \psi(x, y) \) be sufficiently smooth. Then
\[
\psi^i(x, y) = x^i + y^i + b^i_{jk} x^j y^k + \text{(terms of order) } \geq 3
\]
\[
b^i_{jk} = \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \bigg|_{x=y=0}
\]

Let \( \xi, \eta \in T \), and their components in terms of \( x^i \) are \( \xi^i \) and \( \eta^i \).

**Definition.** The **commutator** \([\xi, \eta] \in T\) is defined by
\[
[\xi, \eta]^i = c^i_{jk} \xi^j \eta^k, \quad c^i_{jk} \equiv b^i_{jk} - b^i_{kj}
\]

It has 3 basic properties

1. \([, ,] \) is a bilinear operation on the \( n \)-dim vector space \( T \)
2. Skew-symmetry: \([\xi, \eta] = -[\eta, \xi] \)
3. Jacoby’s identity: \([[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0 \)

**Proof of 3.** Blackboard
Definition. A Lie algebra is a vector space \( G \) over a field \( F \) with a bilinear operation \([\cdot, \cdot] : G \times G \rightarrow G\) which is called a commutator or a Lie bracket, such that the following axioms are satisfied:

- It is skew symmetric: \([x, x] = 0\) which implies \([x, y] = -[y, x]\) for all \(x, y \in G\)
- It satisfies the Jacobi Identity: \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\)

Thus, the tangent space of a Lie group \( G \) at the identity is with respect to the commutator operation a Lie algebra called the Lie algebra of the Lie group \( G \).

Let \( e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n} \) be the standard basis vectors of \( T \) in terms of the coordinates \( x^1, \ldots, x^n \).

Let us multiply \([\xi, \eta]^i = c^i_{jk} \xi^j \eta^k\) by \( e_i \) and take the sum over \(i\)

\([\xi, \eta] = c^i_{jk} \xi^j \eta^k e_i\)

Let us choose \(\xi = e_j, \eta = e_k\). Taking into account that the components of the standard basis vector are

\((e_m)^n = \delta^m_n\)

we get

\([e_j, e_k] = c^i_{jk} e_i\)

Definition. The constants \(c^i_{jk}\) which determine the commutation operation on a Lie algebra, and which are skew-symmetric in \(j, k\), are called the structure constants of the Lie algebra.
### 9.2 One-parameter subgroups and canonical coordinates

**Definition.** A one-parameter subgroup of a Lie group $G$ is defined to be a parametric curve $F(t)$ on the manifold $G$ such that

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F(t)^{-1}$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = \frac{dF(t + \epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \left( F(t)F(\epsilon) \right) \bigg|_{\epsilon=0} = F(t) \frac{dF(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0}$$

Hence,

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F(t)^{-1}\dot{F}(t) = \dot{F}(0),$$

i.e. the induced action of left multiplication by $F(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0) = \text{const} \in T$.

Conversely, $\forall A \in T$, the equation

$$F(t)^{-1}\dot{F}(t) = A$$

is satisfied by a unique one-parameter subgroup $F(t)$ of $G$.

If $G$ is a matrix group then $F(t) = \exp At$.

We will use this notation for arbitrary Lie groups.
Let’s discuss how $F(t)$ and the push-forward map look like in a neighbourhood of $U_0$.

Let $F(t) \in U_0$ have local coordinates $f^1(t), \ldots, f^n(t)$. Since $F(t)$ is a one-parameter subgroup the functions $f^i(t)$ satisfy

$$f^i(0) = 0, \quad f^i(t_1 + t_2) = \psi^i(f(t_1), f(t_2)), \quad f^i(-t) = \varphi^i(f(t))$$

Consider the map $G \mapsto G$ given by the left multiplication by $F(t)$

$$x \mapsto y = F(t) x, \quad x, y \in G$$

If the local coordinates of $x$ and $y$ are $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ then the left multiplication takes the form

$$x \mapsto y : y^i = \psi^i(f(t), x)$$

The corresponding push-forward map induced by the left multiplication is

$$F_*(t) : \xi^i \mapsto \eta^j = \frac{\partial \psi^i(f(t), x)}{\partial x^j} \xi^j, \quad \xi \in T_xG, \quad \eta \in T_yG$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = (\dot{f}^1(t), \ldots, \dot{f}^n(t)),$$

where

$$\dot{f}^i(t) = \frac{df^i(t + \epsilon)}{d\epsilon}\big|_{\epsilon=0} = \frac{d\psi^i(f(t), f(\epsilon))}{d\epsilon}\big|_{\epsilon=0} = \frac{\partial \psi^i(f(t), x)}{\partial x^j}\big|_{x=f(t)} \cdot \dot{f}^j(0)$$

Thus, $F_*(t)$ sends $\dot{F}(0)$ to $\dot{F}(t)$. Similarly, we get

$$\dot{f}^i(0) = \frac{df^i(-t + \epsilon)}{d\epsilon}\big|_{\epsilon=t} = \frac{d\psi^i(f(-t), f(\epsilon))}{d\epsilon}\big|_{\epsilon=t} = \frac{\partial \psi^i(\varphi(f(t)), x)}{\partial x^j}\big|_{x=f(t)} \cdot \dot{f}^j(t)$$

and therefore $F_*(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0)$. 
**Definition.** For each $h \in G$ the transformation $G \mapsto G$ defined by $g \mapsto hgh^{-1}$ is called the **inner automorphism** of $G$ determined by $h$.

Any inner automorphism does not move $g_0$, $hg_0h^{-1} = g_0$, and therefore the push-forward (induced linear) map of the tangent space $T$ to $G$ at $g_0$ is a linear transformation of $T$ denoted by (or by Adj)

$$\text{Ad} (h) : T \mapsto T$$

It satisfies

1. $\text{Ad} (g_0) = id$ where $id$ is the identity transformation of $T$

2. $\text{Ad} (h_1)\text{Ad} (h_2) = \text{Ad} (h_1 h_2)$ for all $h_1, h_2 \in G$

   because $h_1 h_2 g h_2^{-1} h_1^{-1} = (h_1 h_2) g (h_1 h_2)^{-1}$

3. Choosing $h_1 = h$, $h_2 = h^{-1}$, we get $\text{Ad} (h^{-1}) = \text{Ad} (h)^{-1}$

This means that the map $h \mapsto \text{Ad} (h)$ is a **linear representation** of the group $G$, i.e. a homomorphism to a group of linear transformations

$$\text{Ad} : G \mapsto GL(n, \mathbb{R})$$

where $n = \text{dim} (G)$.

This representation of $G$ is called **adjoint**.

For commutative Lie groups $G$, e.g. $U(1)$, the adjoint representation $\text{Ad}$ is trivial, i.e. $\text{Ad} (h) = 1 \forall h \in G$. 

In terms of local coordinates in a neighbourhood of $U_0$ we get the following. Let us denote the inner automorphism of $G$ determined by $h$ by $AD(h)$

$$AD(h) : g \mapsto hgh^{-1}, \quad g, h \in G$$

The corresponding push-forward map is

$$AD(h)_* : \xi^i \mapsto \eta^i = \frac{\partial \psi^i(\psi(h,x), \varphi(h))}{\partial x^j} \xi^j, \quad \xi \in T_g G, \quad \eta \in T_{hgh^{-1}} G$$

where $g$ has local coordinates $(x^1, \ldots, x^n)$.

If $x = 0$ then $g = g_0$ and both $\xi, \eta \in T_{g_0} G = T$, and we get $Ad(h)$

$$Ad(h) : \xi^i \mapsto \eta^i = \frac{\partial \psi^i(\psi(h,x), \varphi(h))}{\partial x^j} \bigg|_{x=0} \xi^j$$

$$= \frac{\partial \psi^i(z, \varphi(h))}{\partial z^k} \bigg|_{z=h} \frac{\partial \psi^k(h,x)}{\partial x^j} \bigg|_{x=0} \xi^j.$$  \hspace{1cm} (9.14)

This formula can be used to show in particular that

$$Ad(h_1)Ad(h_2) = Ad(h_1h_2)$$
Let $F(t) = \exp At$ be a one-parameter subgroup of a Lie group $G$. Then, $\text{Ad}(F(t))$ is a one-parameter subgroup of $GL(n, \mathbb{R})$, and the vector $\frac{d}{dt}\text{Ad}(F(t))|_{t=0}$ lies in the Lie algebra $gl(n, \mathbb{R}) \cong \text{Mat}(n, \mathbb{R})$ of the group $GL(n, \mathbb{R})$ and can be regarded as a linear operator.

This operator is denoted by $\text{ad} A$ and is given by

$$\text{ad} A : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad B \mapsto [A, B], \quad B \in T \cong \mathbb{R}^n$$

The formula is obtained by using (9.14) where we replace $\xi \rightarrow B$, $h \rightarrow f(t)$, differentiate with respect to $t$, set $t = 0$ and use that $\dot{f}^i(0) = A^i$ (prove the formula for $\text{ad} A$).

One-parameter subgroups can be used to define so-called canonical coordinates in a neighbourhood of the identity of a Lie group $G$.

Let $A_1, \ldots, A_n$ form a basis for the Lie algebra $T$.

$\forall A = \sum_i A_i x^i \in T \exists$ a one-parameter group $F(t) = \exp At$.

To the point $F(1) = \exp A$ we assign as coordinates the coefficients $x^1, \ldots, x^n$ which gives us a system of coordinates in a sufficiently small neighbourhood of $g_0 = 1 \in G$.

These are called the canonical coordinates of the first kind.
Another system of coordinates is obtained by introducing $F_i(t) = \exp A_i t$ and representing a point $g$ sufficiently close to $g_0$ as

$$g = F_1(t_1) F_2(t_2) \cdots F_n(t_n)$$

for small $t_1, \ldots, t_n$.

Assigning coordinates $x^1 = t_1, \ldots, x^n = t_n$ to the point $g$, we get the canonical coordinates of the second kind.

**Theorem 3.1.1.** If the functions $\psi^i(x, y)$ defining the multiplication of points $x, y$ of a Lie group $G$ are real analytic (representable by power series) then in some neighbourhood of $g_0 \in G$ the structure of the Lie algebra of $G$ determines the multiplication in $G$.

**Proof.** See the textbook.
Definition 3.1.3. A Lie algebra $\mathcal{G} = \{\mathbb{R}^n, c_{jk}^i\}$ is said to be \textit{simple} if it is noncommutative and has no proper ideals, i.e. subspaces $\mathcal{I} \neq \mathcal{G}, 0$ for which $[\mathcal{I}, \mathcal{G}] \subset \mathcal{I}$, and \textit{semisimple} if $\mathcal{G} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_k$ where the $\mathcal{I}_j$ are ideals which are simple as Lie algebras.

These ideals are pairwise commuting $[\mathcal{I}_i, \mathcal{I}_j] = 0$ for $i \neq j$.

A Lie group is defined to be simple or semisimple according to its Lie algebra.

**Definition.** The \textit{Killing form} on an arbitrary Lie algebra $\mathcal{G}$ is defined by

$$\langle A, B \rangle = -\text{tr}(\text{ad } A \text{ ad } B)$$

**Theorem 3.1.4.**

(i) If the Lie algebra $\mathcal{G}$ of a Lie group $G$ is simple, then the linear representation $\text{Ad} : G \rightarrow GL(n, \mathbb{R})$ is \textit{irreducible}, i.e. $\mathcal{G}$ has no proper invariant subspaces under the group of inner automorphisms $\text{Ad} (G)$.

(ii) If the Killing form of a Lie algebra is positive definite then the Lie algebra is semisimple.

**Proof.** Blackboard

**Remark.** A stronger result due to Killing and Cartan is

A Lie algebra is semisimple if and only if its Killing form is non-degenerate.
9.3 Linear representations

Definition 3.2.1.  
(i) A linear representation of a group $G$ of dim = $n$ is a homomorphism 

$$
\rho : G \mapsto GL(r, \mathbb{R}) \quad \text{or} \quad \rho : G \mapsto GL(r, \mathbb{C})
$$

from $G$ to a group of real or complex matrices. 

(ii) Given a representation $\rho$ of $G$, the map 

$$
\chi_\rho : G \mapsto \mathbb{R} \quad \text{or} \quad G \mapsto \mathbb{C}
$$

defined by 

$$
\chi_\rho(g) = \text{tr} \rho(g), \quad g \in G
$$

is called the character of the representation $\rho$. 

(iii) A representation $\rho$ of $G$ is said to be irreducible if the vector space $\mathbb{R}^r$ (or $\mathbb{C}^r$) contains no proper subspaces invariant under the matrix group $\rho(G)$. 

Theorem 3.2.2 (Schur’s Lemma).

Let
\[ \rho_i : G \mapsto GL(r_i, \mathbb{R}) , \quad i = 1, 2 \]
be two irreps of a group \( G \). If \( A : \mathbb{R}^{r_1} \mapsto \mathbb{R}^{r_2} \) is a linear transformation changing \( \rho_1 \) into \( \rho_2 \), i.e. satisfying
\[ A\rho_1(g) = \rho_2(g)A , \quad \forall g \in G \]
then either \( A \) is the zero transformation or else a bijection (in which case \( r_1 = r_2 \)).

**Proof.** Blackboard
If $G$ is a Lie group and a representation $\rho : G \mapsto GL(r, \mathbb{R})$ is a smooth map, then the push-forward map $\rho_*$ is a linear map from the Lie algebra $\mathcal{G} = T_{(1)}$ to the space of all $r \times r$ matrices

$$\rho_* : \mathcal{G} \mapsto Mat(r, \mathbb{R})$$

Verify that $\rho_*$ is a representation of the Lie algebra $\mathcal{G}$, i.e. that it is a Lie algebra homomorphism:

1. It is linear

2. It preserves commutators

$$\rho_*[\xi, \eta] = [\rho_*\xi, \rho_*\eta]$$
Definition. A representation
\[ \rho : G \mapsto GL(r, \mathbb{R}) \text{ or } \rho : G \mapsto GL(r, \mathbb{C}) \]
is called \textit{faithful} if it is one-to-one, i.e. if its kernel is trivial
\[ \rho(g) \neq I \text{ unless } g = g_0 \]

If a Lie group has a faithful representation then it can be realised as a matrix Lie group.

Any matrix Lie group obviously has a faithful representation.

However, not every Lie group can be realised as a matrix Lie group.

One such an example is the group \( \tilde{SL}(2, \mathbb{R}) \) of all transformations of the real line of the form
\[ x \to x + 2\pi a + \frac{1}{i} \ln \frac{1 - ze^{-ix}}{1 - \bar{z}e^{ix}}, \]
where \( x \in \mathbb{R}, a \in \mathbb{R}, z \in \mathbb{C}, |z| < 1 \) and \( \ln \) is the main branch of the natural logarithmic function, i.e. the continuous branch determined by \( \ln 1 = 0 \).

\( \tilde{SL}(2, \mathbb{R}) \) is a universal covering group of \( SL(2, \mathbb{R}) \), i.e. it has the same Lie algebra and it is simply connected.
10 Homogeneous Spaces

10.1 Action of a group on a manifold

**Definition 5.1.1.** We say that a Lie group $G$ is represented as a group of transformations of a manifold $M$, or has a left action on $M$ if

1. there is associated with each of its elements $g$ a diffeomorphism from $M$ to itself

$$x \mapsto T_g(x), \quad x \in M,$$

such that $T_{gh} = T_g T_h$ for all $g, h \in G$

2. $T_g(x)$ depends smoothly on the arguments $g, x$, i.e. the map

$$(g, x) \mapsto T_g(x)$$

is a smooth map from $G \times M$ to $M$.

The Lie group $G$ is said to have a right action on $M$ if the above definition is valid with $T_g T_h = T_{gh}$ replaced by $T_g T_h = T_{hg}$.

**Example 1.** Let $M = G$. Is the action below left or right?

1. $h \mapsto T_g(h) = gh, \quad h \in G$
2. $h \mapsto T_g(h) = hg, \quad h \in G$
3. $h \mapsto T_g(h) = g^{-1}h, \quad h \in G$
4. $h \mapsto T_g(h) = hg^{-1}, \quad h \in G$

**Example 2.** Any group of real $n \times n$ matrices acts on $\mathbb{R}^n$, e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$
**Definition.** The action of a group $G$ on $M$ is said to be *transitive* if for every two points $x, y$ of $M$ there exists an element $g$ of $G$ such that $T_g(x) = y$.

**Definition 5.1.2.** A manifold on which a Lie group acts transitively is called a *homogeneous* space of the Lie group.

In particular, $G$ is a homogeneous space for itself, e.g. as $h \mapsto T_g(h) = gh$, $h \in G$. $G$ is called the *principal* homogeneous space.

**Definition.** Let $x$ be any point of a homogeneous space $M$ of a Lie group $G$. The *isotropy* group (or *stationary* group) $H_x$ of the point $x$ is the stabiliser of $x$ under the action of $G$:

$$H_x = \{ h | T_h(x) = x \} .$$

**Lemma 5.1.3.** All isotropy groups $H_x$ of points $x$ of a homogeneous space are isomorphic.

**Proof.** Let $x, y$ be any two points of the homogeneous space and $g$ be an element of the Lie group such that $T_g(x) = y$. The map $H_x \mapsto H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism.

Prove that $ghg^{-1}$ is an element of $H_y$
Theorem 5.1.4. There is a one-to-one correspondence between the points of a homogeneous space $M$ of a group $G$, and the left cosets $gH$ of $H$ in $G$, where $H$ is the isotropy group and $G$ acts on the left.

**Proof.** Recall that if $G$ is a group, and $H$ is a subgroup of $G$, and $g \in G$, then $gH = \{gh : h \in H\}$ is the left coset of $H$ in $G$ with respect to $g$, while $Hg = \{hg : h \in H\}$ is the right coset.

Let $x_0$ be any point of $M$. Then, we put in correspondence to each left coset $gH_{x_0}$ the point $T_g(x_0) \in M$. This correspondence is independent of the choice of representative of the coset, one-to-one, and onto.

*Prove it*

Thus, we can write

$$M \cong G/H$$
10.2 Examples of Homogeneous Spaces

1. Sphere:
\[ S^n \cong O(n + 1)/O(n) \cong SO(n + 1)/SO(n) \]

2. **Real projective space:**
\[ \mathbb{R}P^n \cong O(n + 1)/(O(1) \times O(n)) \]

3. **Torus:**
\[ T^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/Z^n \]

4. **Stiefel manifolds:**
\[ V_{n,k} \cong O(n)/O(n - k) \cong SO(n)/SO(n - k) \]

5. **Real Grassmanian manifolds:**
\[ G_{n,k} \cong O(n)/(O(k) \times O(n - k)) \]

6. **Homogeneous spaces for** \( U(n) \)
   
   (a) **Sphere:**
\[ S^{2n+1} \cong U(n + 1)/U(n) \cong SU(n + 1)/SU(n) \]

   (b) **Complex projective space:**
\[ \mathbb{C}P^n \cong U(n + 1)/(U(1) \times U(n)) \]

   (c) **Complex Grassmanian manifolds:**
\[ G^{C}_{n,k} \cong U(n)/(U(k) \times U(n - k)) \]
11 Vector Bundles on a Manifold

11.1 Tangent bundle $T(M)$

**Definition.** The *tangent bundle* $T(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T(M)$ are the pairs $(x, \xi)$, $x \in M$ and $\xi \in T_x M$

2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U_q^T$ of $T(M)$ is the set of all pairs $(x, \xi)$ where

$$x = (x^1_q, \ldots, x^n_q) \in U_q \quad \text{and} \quad \xi = \xi^i_q \frac{\partial}{\partial x^i_q} \in T_x M$$

with the local coordinates

$$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, \xi^1_q, \ldots, \xi^n_q) = (x^i_q, \xi^i_q)$$

**Proposition 7.1.1.** The tangent bundle $T(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U_q^T \cap U_p^T$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, \xi^i_p) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^i_p}{\partial x^k_q} \xi^k_p)$$

The Jacobian matrix is

$$\left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = \begin{pmatrix} A & 0 \\ H & A \end{pmatrix}, \quad A = \left( \frac{\partial x^i_p}{\partial x^j_q} \right), \quad H = \left( \frac{\partial^2 x^i_p}{\partial x^j_q \partial x^k_q} \xi^k_p \right)$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = (\det A)^2 > 0$$
11.2 Cotangent bundle $T^*(M)$

**Definition.** The **cotangent bundle** $T^*(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T^*(M)$ are the pairs $(x, p)$, $x \in M$, and $p$ is a co-vector at the point $x$: $p \in T^*_x M$

2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U^T_q$ of $T^*(M)$ is the set of all pairs $(x, p)$ where

$$x = (x^1_q, \ldots, x^n_q) \in U_q \quad \text{and} \quad p = p_q dx^i_q \in T^*_x M$$

with the local coordinates

$$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, p_{q_1}, \ldots, p_{q_n}) = (x^i_q, p_{qi})$$

**Proposition.** The cotangent bundle $T^*(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U^T_q \cap U^T_p$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, p_{pi}) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^k_q}{\partial x^i_p} p_{qk})$$

The Jacobian matrix is

$$\left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = \begin{pmatrix} A & 0 \\ \tilde{H} & A^{-1} \end{pmatrix}, \quad A = \left( \frac{\partial x^i_p}{\partial x^j_q} \right), \quad \tilde{H} = \left( \frac{\partial^2 x^k_q}{\partial x^j_q \partial x^i_p} p_{qk} \right)$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = 1 > 0$$
The existence of a metric $g_{ij}$ on $M$ gives rise to a map

$$T(M) \mapsto T^*(M) : (x^i, \xi^i) \mapsto (x^i, g_{ij} \xi^j)$$

Since $\omega = p_i dx^i$, a differential one-form on $M$, is invariant under a change of coordinates of $T^*(M)$, it is a differential form on $T^*(M)$.

Its differential

$$\Omega = d\omega = dp_i \wedge dx^i$$

is a nondegenerate closed, $d\Omega = 0$, 2-form on $T^*(M)$.

Thus, $T^*(M)$ is a symplectic manifold, i.e. it is equipped with a closed nondegenerate 2-form.
11.3 Normal vector bundle on a submanifold

Let $M$ be an $n$-dim Riemann manifold with metric $g_{ij}$, and let $N$ be a smooth $k$-dim submanifold of $M$. We assume that $N$ is defined by a non-singular system of $(n - k)$ equations.

Recall that the scalar product of two vectors $\xi, \eta \in T_xM$ is given by

$$\langle \xi, \eta \rangle = g_{ij}\xi^i\eta^j$$

Let $x \in N$, and let $\nu \in T_xM$, and let $\nu$ be orthogonal to $N$ at $x$, i.e. orthogonal to the tangent space to $N$ at $x$, which is a $k$-dim subspace of $T_xM$. So, $\nu$ form a $(n - k)$-dim subspace of $T_xM$.

**Definition and Theorem.** The normal vector bundle $\nu_M(N)$ on the submanifold $N$ in $M$ is an $n$-dim submanifold of $T(M)$ defined as

1. The points of $\nu_M(N)$ are the pairs $(x, \nu)$, $x \in N$, $\nu \in T_xM$, $\nu \perp N$

2. Given a chart $U$ of $M$ with suitable local coordinates

   $y^i, i = 1, \ldots, n, N$ is defined in $U$ by the equations

   $y^{k+1} = 0, \ldots y^n = 0,$

   and $y^1, \ldots, y^k$ serve as local coordinates on $N$.

3. The normal bundle $\nu_M(N)$ is determined as an $n$-dim submanifold of $T(M)$ by the equations

   $y^{k+1} = 0, \ldots y^n = 0, \quad g_{ij}\nu^j = 0, \quad i = 1, \ldots, k$
Examples.

1. Let $M = \mathbb{R}^n$, and let $N$ be defined by the nonsingular system

$$f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y = (y^1, \ldots, y^n)$$

where $y^i$ are Euclidean coordinates on $\mathbb{R}^n$.

Then, the vectors $\nabla f_1, \ldots, \nabla f_{n-k}$, are at each point of $N$ perpendicular to $N$ and linearly independent.

Hence,

$$\nu_{\mathbb{R}^n}(N) \cong N \times \mathbb{R}^{n-k}$$
2. More generally, if $N$ is defined as a submanifold of $M$ by a non-singular system

$$f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y \in M$$

then at each point $x \in N$ the vector fields

$$e^i_a(x) = g^{ik}(y) \frac{\partial f_a}{\partial y^k} \bigg|_{y=y(x)}, \quad g^{ik}(y)g_{kj}(y) = \delta^i_j, \quad a = 1, \ldots, n-k$$

are linear independent, and for each $\xi \in T_xN$ they satisfy

$$g_{ij}e^i_a(x)\xi^j(x) = \frac{\partial f_a}{\partial y^j} \xi^j(x) \bigg|_{y=y(x)} = 0$$

Thus, they are orthogonal to $N$ and any vector normal to $N$ at $x \in N$ has the form

$$\nu = \nu^a e_a(x)$$

The correspondence

$$(x, \nu) \mapsto (x, \nu^1, \ldots, \nu^{n-k})$$

is then a diffeomorphism

$$\nu_M(N) \cong N \times \mathbb{R}^{n-k}$$
3. In particular, if $N$ is the boundary of $\partial A$ of a manifold $A$ with boundary defined by an inequality $f(y) \leq 0$ then $\partial A$ is defined by the single equation $f(y) = 0$, and the normal bundle to the boundary decomposes as a direct product

$$\nu_M(\partial A) \cong \partial A \times \mathbb{R}$$

The bundles we considered are particular cases of *smooth fibre bundles*, see the textbook.