Contents

1 Definition of Manifold ................................................. 2

2 Elements of Topology ................................................ 8

3 Mappings of Manifolds. Tensors on Manifolds .................. 15

4 Algebraic Operations on Tensors (vol 1, section 17) ............ 21

5 Tensors of type \((0, k)\) (vol 1, section 18) ...................... 26
  5.1 Co-vectors: Tensors of type \((0, 1)\) .......................... 26
  5.2 Tensors of type \((0, 2)\) ............................................. 27
  5.3 Skew-symmetric Tensors of type \((0, k)\) ...................... 29

6 The behaviour of Tensors under Mappings (vol 1, section 22) 33
  6.1 Tensors of type \((k, 0)\) .............................................. 33
  6.2 Tensors of type \((0, k)\) .............................................. 35

7 Embeddings and Immersions of Manifolds ....................... 39

8 Surfaces in Euclidean space ........................................ 42
  8.1 Surfaces as Manifolds ............................................ 42
  8.2 Surfaces can be oriented ......................................... 44
  8.3 Transformation Groups as Surfaces .......................... 49

9 Projective Spaces ...................................................... 52
  9.1 Real Projective Space ............................................. 52
  9.2 Quaternions, \(SU(2), SO(3), \mathbb{R}P^3\) ...................... 53
  9.3 \(\mathbb{R}P^n\) as a manifold ........................................ 58
  9.4 Complex Projective Space ...................................... 60

10 Elements of the Theory of Lie Groups ............................ 62
  10.1 Neighbourhood of the Identity Element of a Lie Group .... 62
  10.2 One-parameter subgroups and canonical coordinates ........ 65
  10.3 Linear representations ......................................... 72

11 Homogeneous Spaces ................................................ 76
  11.1 Action of a group on a manifold ............................... 76
  11.2 Examples of Homogeneous Spaces ............................ 79

12 Vector Bundles on a Manifold ...................................... 80
  12.1 Tangent bundle \(T(M)\) .......................................... 80
  12.2 Cotangent bundle \(T^*(M)\) ..................................... 81
  12.3 Normal vector bundle on a submanifold ...................... 83
Differentiable Manifolds

1 Definition of Manifold

$\mathbb{R}^n$ is an $n$-dim Cartesian or vector or linear space.

**Definition.** A region, or a region without boundary, ("open set") is a set $D$ of points in $\mathbb{R}^n$ such that together with each point $P_0$, $D$ also contains all points sufficiently close to $P_0$, i.e.

$$\forall P_0 = (x_0^1, \ldots, x_0^n) \in D \ \exists \epsilon > 0 :$$

all points $P = (x^1, \ldots, x^n)$ satisfying $|x^i - x_0^i| < \epsilon$, $i = 1, \ldots, n$ also lie in $D$. 
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**Definition.** A *region with boundary* is obtained from a region $D$ by adjoining all boundary points (i.e. points not in $D$, yet having points of $D$ arbitrarily close to them). The *boundary* of a region is just the set of boundary points.
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all points $P = (x^1, \ldots, x^n)$ satisfying $|x^i - x_0^i| < \epsilon$, $i = 1, \ldots, n$ also lie in $D$.

**Definition.** A region with boundary is obtained from a region $D$ by adjoining all boundary points (i.e. points not in $D$, yet having points of $D$ arbitrarily close to them). The boundary of a region is just the set of boundary points.

**Definition.** $n$-dim *Euclidean* space is $\mathbb{R}^n$ with the distance $l$ between any two points given by

$$l^2 = \sum_{i=1}^{n} (x^i - y^i)^2$$
**Definition 1.1.1.** A differentiable \( n \)-dimensional manifold is a set \( M \) (whose elements we call “points”) together with the following structure on it. The set \( M \) is the union of a finite or countably infinite collection of subsets \( U_q \) with the following properties

(i) Each subset \( U_q \) has defined on it coordinates \( x_q^\alpha, \alpha = 1, \ldots, n \) (called local coordinates) by virtue of which \( U_q \) is identifiable with a region of Euclidean \( n \)-space \( \mathbb{R}^n \) with Euclidean coordinates \( x_q^\alpha \). The \( U_q \) with their coordinate systems are called *charts* or *local coordinate neighbourhoods*.

(ii) Each non-empty intersection \( U_p \cap U_q \) of a pair of charts thus has defined on it two coordinate systems, the restrictions of \((x_p^\alpha)\) and \((x_q^\alpha)\). It is required that under each of these coordinatisations the intersection \( U_p \cap U_q \) is identifiable with a region of \( \mathbb{R}^n \), and that each of these coordinate systems be expressible in terms of the other in a one-to-one differentiable manner. Thus, if the *transition* functions from \( x_q^\alpha \) to \( x_p^\alpha \) and back are given by

\[
\begin{align*}
x_p^\alpha &= x_p^\alpha(x_q^1, \ldots, x_q^n), & \alpha = 1, \ldots, n, \\
x_q^\alpha &= x_q^\alpha(x_p^1, \ldots, x_p^n), & \alpha = 1, \ldots, n,
\end{align*}
\]

then in particular the *Jacobian* \( \det(\partial x_p^\alpha/\partial x_q^\beta) \) is nonzero on \( U_p \cap U_q \).

The general *smoothness* class of the transition functions for all intersecting pairs \( U_p, U_q \) is called the smoothness class of the manifold \( M \) with its accompanying atlas of charts \( U_q \).
Example 1. Any Euclidean space of regions is a manifold.

Example 2. A region of complex space $\mathbb{C}^n$ can be regarded as a region of $\mathbb{R}^{2n} \implies \mathbb{C}^n$ is a manifold.

Example 3. A 2-sphere $S^2$ is a manifold.

Example 4. A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.
Example 1. Any Euclidean space of regions is a manifold.

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Example 4. A circle $S^1$, and in general an $n$-sphere $S^n$ is a manifold.

Example 5. Given two manifolds $M = \cup_q U_q$, $N = \cup_p V_p$ we construct their direct product $M \times N$ as follows:

The points of the manifold $M \times N$ are ordered pairs $(m, n)$, and covering by local coordinate neighbourhoods is given by

$$M \times N = \cup_{q,p} U_q \times V_p$$

where if $x_{q}^{\alpha}$ are the coordinates on $U_q$ and $y_{p}^{\beta}$ on $V_p$ then the coordinates on $U_q \times V_p$ are $(x_{q}^{\alpha}, y_{p}^{\beta})$.

E.g. $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times S^1$, $S^1 \times \mathbb{R}$, $S^1 \times S^1$, $\mathbb{R}^m \times \mathbb{R}^n$. 
2 Elements of Topology

The definition of manifold is very general. To restrict it we need some basic concepts of topology.

**Definition.** A topological space is a set $X$ (of “points”) of which certain subsets, called the *open sets* of the topological space, are distinguished. These open sets have to satisfy:

(a) the intersection of any two (and hence of any finite collection) of them should again be an open set;

(b) the union of any collection of open sets must again be open;

(c) the empty set and the whole set $X$ must be open.

The complement of any open set is called a *closed set* of the topological space.

In Euclidean space $\mathbb{R}^n$ the “Euclidean topology” is the usual one where the open sets are the open regions.
**Definition.** A topological space is a set \( X \) (of “points”) of which certain subsets, called the **open sets** of the topological space, are distinguished. These open sets have to satisfy:

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The complement of any open set is called a **closed set** of the topological space.

In Euclidean space \( \mathbb{R}^n \) the “Euclidean topology” is the usual one where the open sets are the open regions.

**Definition.** Given any subset \( A \in \mathbb{R}^n \), the **induced topology** on \( A \) is that with open sets the intersections \( A \cap U \) where \( U \) ranges over all open sets of \( \mathbb{R}^n \).

**Definition.** A map \( f : X \mapsto Y \) of one topological space to another is **continuous** if the complete inverse image \( f^{-1}(U) \) of every open set \( U \subseteq Y \) is open in \( X \).

**Definition.** Two topological spaces are **topologically equivalent** or **homeomorphic** if there is a one-to-one and onto map between them such that both it and its inverse are continuous.
**Definition 1.1.2.** The topology on a manifold $M$ is given by the following specification of the open sets.

In every local coordinate neighbourhood $U_q$ the open regions are to be open in the topology on $M$; the totality of open sets of $M$ is then obtained by admitting as open also arbitrary unions of countable collections of such regions, i.e. by closing under countable unions.

With this topology the continuous maps of a manifold $M$ turn out to be those which are continuous in the usual sense on each local coordinate neighbourhood $U_q$.

Any open subset $V$ of $M$ inherits, i.e. has induced on it, the structure of a manifold: $V = \bigcup_q V_q$ where $V_q = V \cap U_q$. 
**Definition.** A *metric space* is a set which comes equipped with a “distance function”, i.e. a real-valued function $\rho(x, y)$ defined on pairs $x, y$ of its elements (“points”), and having the following properties

(i) Symmetry: $\rho(x, y) = \rho(y, x),$

(ii) Positivity: $\rho(x, x) = 0,$ $\rho(y, x) > 0$ if $x \neq y,$

(iii) The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y),$

**Example.** $n$-dim Euclidean space is a metric space with

$$\rho(x, y) = \sqrt[n]{\sum_{\alpha=1}^{n} (x^{\alpha} - y^{\alpha})^2}.$$ 

A metric space is topologised by taking as its open sets the unions of arbitrary collections of “open balls” where by open ball with centre $x_0$ and radius $\epsilon$ we mean the set of all points $x$ satisfying $\rho(x, x_0) < \epsilon.
**Definition.** A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centres at $x, y$ do not intersect.

All topological spaces we consider will be Hausdorff. In particular our manifolds will be Hausdorff spaces.
**Definition.** A topological space $X$ is said to be *compact* if every countable collection of open sets covering $X$ contains a finite subcollection already covering $X$.

If $X$ is a metric space then compactness is equivalent to the condition that from every sequence of points of $X$ a convergent subsequence can be selected.

**Definition.** A topological space is connected if any two points can be joined by a continuous path.
**Definition 1.1.3.** A manifold $M$ is said to be *oriented* if for every pair $U_p, U_q$ of intersecting local coordinate neighbourhoods the Jacobian $J_{pq} = \det(\partial x_\alpha^p / \partial x_\beta^q)$ of the transition functions is positive.

Euclidean $\mathbb{R}^n$ and $S^2$ are oriented.

**Definition 1.1.4.** We say that the coordinate systems $x$ and $y$ define the *same orientation* of $\mathbb{R}^n$ if $J > 0$ and *opposite orientations* if $J < 0$.

Euclidean $\mathbb{R}^n$ possesses two possible orientations, and any *connected* oriented manifold has exactly two orientations.
3 Mappings of Manifolds. Tensors on Manifolds

Let $M = \bigcup_p U_p$ with coordinates $x_\alpha^p$, and $N = \bigcup_q V_q$ with coordinates $y_\beta^q$ be two manifolds of dim $m$ and $n$.

**Definition 1.2.1.** A mapping $f : M \mapsto N$ is said to be smooth of smoothness class $k$ if for all $p, q$ for which $f$ determines functions $y_\beta^q(x_1^p, \ldots, x_m^p) = f(x_1^p, \ldots, x_m^p)_q$, these functions are, where defined, smooth of smoothness class $k$ (i.e. all their partial derivatives up to those of $k$th order exist and are continuous).

The smoothness class of $f$ cannot exceed the maximum class of the manifolds.

If $N = \mathbb{R}$ then $f$ is a real-valued function of the points of $M$.

A smooth mapping *may not* be defined on the whole manifold $M$.

E.g. each local coordinate $x_\alpha^p$ for fixed $p$ and $\alpha$ is a real-valued function on $M$ defined only on the region $U_p$.

**Definition 1.2.2.** The manifolds $M$ and $N$ are said to be *smoothly equivalent* or *diffeomorphic* if there is a one-to-one and onto map $f$ such that both $f : M \mapsto N$ and $f^{-1} : N \mapsto M$ are smooth of some class $k \geq 1$.

Since $f^{-1}$ exists then $J_{pq} = \det(\partial y_\beta^q / \partial x_\alpha^p) \neq 0$ wherever it is defined.

We always assume that the smoothness class of any manifolds and mappings are sufficiently high for our aims.
Let $x = x(\tau)$, $a \leq \tau \leq b$, be a curve segment on a manifold $M \ni x(\tau)$. In $U_p$ with $x_p^\alpha$ it is described by the parametric equations

$$x_p^\alpha = x_p^\alpha(\tau), \quad \alpha = 1, \ldots, m,$$

and in $U_p$ its velocity vector (which is tangent to the curve) is

$$\dot{x} = (\dot{x}_p^1, \ldots, \dot{x}_p^m).$$

In $U_p \cap U_q$ we have two representations $x_p^\alpha(\tau)$ and $x_q^\beta(\tau)$ of the curve and

$$x_p^\alpha(x_q^1(\tau), \ldots, x_q^m(\tau)) = x_p^\alpha(\tau).$$

Thus, velocities in the two systems are related as

$$\dot{x}_p^\alpha = \sum_{\beta=1}^m \frac{\partial x_p^\alpha}{\partial x_q^\beta} \dot{x}_q^\beta \equiv \frac{\partial x_p^\alpha}{\partial x_q^\beta} \dot{x}_q^\beta \quad \forall \alpha,$$

where we use Einstein’s summation rule:

- each index can appear at most twice in any term;
- repeated indices are implicitly summed over;
- one repeated index must be upper and the other one must be lower.
Definition 1.2.3. A tangent vector to an $m$-dim manifold $M$ at an arbitrary point $x$ is represented in terms of local coordinates $x^\alpha_p$ by an $m$-tuple $(\xi^\alpha)$ of components which are linked to the components in terms of any other system $x^\beta_q$ of local coordinates as

$$
\xi^\alpha_p = \left( \frac{\partial x^\alpha_p}{\partial x^\beta_q} \right)_x \xi^\beta_q \quad \forall \alpha,
$$

(3.2)

The set of all tangent vectors to an $m$-dim manifold $M$ at a point $x$ forms an $m$-dim vector space $T_x = T_x M$, the tangent space to $M$ at the point $x$.

Thus, the velocity vector at $x$ of any smooth curve on $M$ through $x$ is a tangent vector to $M$ at $x$.

From (3.2) one sees that for any choice of local coordinates $x^\alpha$ in a neighbourhood of $x$, the operators $\frac{\partial}{\partial x^\alpha}$, operating on real-valued functions on $M$, may be thought of as forming a basis $e^\alpha = \frac{\partial}{\partial x^\alpha}$ for the tangent space $T_x$.

$$
(3.2) \quad \Rightarrow \quad \xi^\alpha_p \frac{\partial}{\partial x^\alpha_p} = \xi^\beta_q \frac{\partial}{\partial x^\beta_q}.
$$
Definition. A smooth map $f$ from $M$ to $N$ gives rise for each $x$ to a \textit{push-forward} or \textit{an induced linear} map of tangent spaces

$$f_* : T_xM \mapsto T_{f(x)}N,$$

defined as sending the velocity vector at $x$ of any smooth curve $x = x(\tau)$ on $M$ to the velocity vector at $f(x)$ to the curve $f(x(\tau))$ on $N$.

In terms of local coordinates $x^\alpha$ in a neighbourhood of $x \in M$, and $y^\beta$ in a neighbourhood of $f(x) \in N$ the map $f$ is written as

$$y^\beta = f^\beta(x^1, \ldots, x^m), \quad \beta = 1, \ldots, n,$$

and the push-forward map $f_*$ as

$$\xi^\alpha \mapsto \eta^\beta = \frac{\partial f^\beta}{\partial x^\alpha} \xi^\alpha$$

For a real-valued function $f : M \mapsto \mathbb{R}$, the push-forward map $f_*$ corresponding to each $x \in M$ is a real-valued linear function on the tangent space to $M$ at $x$

$$\xi^\alpha \mapsto \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha,$$

and it is represented by the gradient of $f$ at $x$, and is a co-vector or one-form. Thus, $f_*$ can be identified with the differential $df$. In particular

$$dx^\alpha_p : \xi^\alpha \mapsto \eta = \xi^\alpha_p.$$
Definition 1.2.4. A Riemann metric on a manifold $M$ is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local coordinates of the points. Thus, at each point $x = (x_1^p, \ldots, x_m^p)$ of each chart $U_p$, the metric is given by a symmetric matrix $(g^{(p)}_{\alpha\beta}(x_1^p, \ldots, x_m^p))$, and determines a symmetric scalar product of pairs of tangent vectors at the point $x$

$$\langle \xi, \eta \rangle = g^{(p)}_{\alpha\beta} \xi_\alpha^p \eta_\beta^p = \langle \eta, \xi \rangle, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be coordinate-independent

$$g^{(p)}_{\alpha\beta} \xi_\alpha^p \eta_\beta^p = g^{(q)}_{\alpha\beta} \xi_\alpha^q \eta_\beta^q$$

and therefore the coefficients $g^{(p)}_{\alpha\beta}$ of the quadratic form transform as

$$g^{(q)}_{\gamma\delta} = \frac{\partial x_\alpha^p}{\partial x_\gamma^q} \frac{\partial x_\beta^p}{\partial x_\delta^q} g^{(p)}_{\alpha\beta}$$

For a pseudo-Riemann metric on $M$ one just requires the quadratic form to be nondegenerate.
**Definition 1.2.4.** A *tensor of type* \((k, l)\) and *rank* \(k + l\) on an \(m\)-dim manifold \(M\) is given in each local coordinate system \(x_\alpha^p\) by a family of functions

\[
(p)T^{i_1\ldots i_k}_{j_1\ldots j_l}(x)
\]

of the point \(x\).

In other local coordinates \(x_\beta^q\) the components \((q)T^s_{t_1\ldots t_l}(x)\) of the same tensor are

\[
(q)T^s_{t_1\ldots t_l} = \frac{\partial x_q^{s_1}}{\partial x_p^{i_1}} \ldots \frac{\partial x_q^{s_k}}{\partial x_p^{i_k}} \frac{\partial x_p^{j_1}}{\partial x_q^{t_1}} \ldots \frac{\partial x_p^{j_l}}{\partial x_q^{t_l}} \cdot (p)T^{i_1\ldots i_k}_{j_1\ldots j_l} \tag{3.3}
\]

Let’s rewrite \((3.3)\) as

\[
(q)T^s_{t_1\ldots t_l} \, dx_q^{t_1} \ldots dx_q^{t_l} \frac{\partial}{\partial x_q^{s_1}} \ldots \frac{\partial}{\partial x_q^{s_k}} = (p)T^{i_1\ldots i_k}_{j_1\ldots j_l} \, dx_p^{j_1} \ldots dx_p^{j_l} \frac{\partial}{\partial x_p^{i_1}} \ldots \frac{\partial}{\partial x_p^{i_k}}
\]
4 Algebraic Operations on Tensors (vol 1, section 17)

1. Permutation of indices. Let $\sigma$ be some permutation of $1, 2, \ldots, l$

$$\sigma = \begin{pmatrix} 1 & \ldots & l \\ \sigma(1) & \ldots & \sigma(l) \end{pmatrix}$$

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

$$\sigma(j_1, \ldots, j_l) = (j_{\sigma_1}, \ldots, j_{\sigma_l})$$

We say that a tensor $\tilde{T}^i_{j_1\ldots j_k}$ is obtained from a tensor $T^i_{j_1\ldots j_k}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$\tilde{T}^i_{j_1\ldots j_k} = T^{i}_{\sigma(j_1\ldots j_k)}$$

Permutations of the upper indices are defined similarly.

Example. $\tilde{T}^i_{j} = T^i_{\sigma(i)} = T^i_{j}$ which is a matrix transposition.
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We say that a tensor $\tilde{T}^{i_1\cdots i_k}_{j_1\cdots j_l}$ is obtained from a tensor $T^{i_1\cdots i_k}_{j_1\cdots j_l}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$\tilde{T}^{i_1\cdots i_k}_{j_1\cdots j_l} = T^{i_1\cdots i_k}_{\sigma(j_1\cdots j_l)}$$

Permutations of the upper indices are defined similarly.

**Example.** $\tilde{T}_{ij} = T_{\sigma(ij)} = T_{ji}$ which is a matrix transposition.

2. **Contraction (taking “traces”).** By the contraction of a tensor $T^{i_1\cdots i_k}_{j_1\cdots j_l}$ of type $(k, l)$ with respect to the indices $i_a, j_b$ we mean the tensor (summation over $n$)

$$\tilde{T}^{i_1\cdots i_{k-1}}_{j_1\cdots j_{l-1}} = T^{i_1\cdots i_{a-1}i_{a+1}\cdots i_k}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_l}$$

of type $(k - 1, l - 1)$.

**Example.** $T^n = \text{tr} T$ of the matrix $T^i_j$. 

22
1. Permutation of indices. Let $\sigma$ be some permutation of $1, 2, \ldots, l$

$$\sigma = \begin{pmatrix} 1 & \ldots & l \\ \sigma(1) & \ldots & \sigma(l) \end{pmatrix}$$

$\sigma$ acts on the ordered $l$-tuple $(j_1, \ldots, j_l)$ as

$$\sigma(j_1, \ldots, j_l) = (j_{\sigma(1)}, \ldots, j_{\sigma(l)})$$

We say that a tensor $\tilde{T}^{i_1\cdots i_k}$ is obtained from a tensor $T^{i_1\cdots i_k}$ by means of a permutation $\sigma$ of the lower indices if at each point of $M$

$$\tilde{T}^{i_1\cdots i_k} = T^{i_{\sigma(1)}\cdots i_{\sigma(l)}}$$

Permutations of the upper indices are defined similarly.

**Example.** $\tilde{T}_{ij} = T_{\sigma(ij)} = T_{ji}$ which is a matrix transposition.

2. Contraction (taking “traces”). By the contraction of a tensor $T^{i_1\cdots i_k}$ of type $(k, l)$ with respect to the indices $i_a, j_b$ we mean the tensor (summation over $n$)

$$\tilde{T}^{i_1\cdots i_{k-1}} = T^{i_1\cdots i_a-1n^a_i+1\cdots i_{k-1}}$$

of type $(k-1, l-1)$.

**Example.** $T^n = \text{tr} T$ of the matrix $T^i_j$.

3. Product of tensors. Given two tensors $T = (T^{i_1\cdots i_p}_{j_1\cdots j_q})$ of type $(p, q)$ and $P = (P^{i_1\cdots i_k}_{j_1\cdots j_l})$ of type $(k, l)$, we define their product to be the tensor $S = T \otimes P$ of type $(p + k, q + l)$ with components

$$S^{i_1\cdots i_{p+k}}_{j_1\cdots j_{q+l}} = T^{i_1\cdots i_p}_{j_1\cdots j_q} P^{i_{p+1}\cdots i_{p+k}}_{j_{q+1}\cdots j_{q+l}}$$

This multiplication is **not commutative** but it is **associative**
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^i_j = \xi^i \eta_j$ of type $(1, 1)$. Contraction $T^i_i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_l \Rightarrow T^{ik}_l = \xi^i A^k_l$ of type $(2, 1)$. Contraction $\eta^k = \xi^i A^k_i$ is a vector, the result of applying the linear transformation $A^k_i$ to the vector.
Lemma. The results of applying the operations 1-3 to tensors are again tensors.

HW: Prove the lemma

Example 1. Vector $\xi^i$, co-vector $\eta_j \Rightarrow$ their tensor product $T^i_j = \xi^i \eta_j$ of type $(1,1)$. Contraction $T^i_i = \xi^i \eta_i$ is a scalar, the scalar product of the vector and co-vector.

Example 2. Vector $\xi^i$, linear operator $A^k_l \Rightarrow T^{ik}_l = \xi^i A^k_l$ of type $(2,1)$. Contraction $\eta^k_l = \xi^i A^i^k$ is a vector, the result of applying the linear transformation $A^k_l$ to the vector.

Example 3. We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows:
Since the gradient $\frac{\partial f}{\partial x^i}$ of a function $f$ is a co-vector, the quantity
\[
\partial_{\xi} f = \xi^i \frac{\partial f}{\partial x^i}
\]
is a scalar called the directional derivative of $f$ in the direction of $\xi$. Thus, an arbitrary vector $\xi$ corresponds to the operator
\[
\partial_{\xi} = \xi^i \frac{\partial}{\partial x^i}
\]
We identify $e_i = \frac{\partial}{\partial x^i}$ with the canonical basis of the tangent space.
5 Tensors of type \((0, k)\) (vol 1, section 18)

These are tensors with lower indices: \(T_{i_1 \ldots i_k}\)

5.1 Co-vectors: Tensors of type \((0, 1)\)

The gradient \(\left(\frac{\partial f}{\partial x^i}\right)\) of a function \(f\) is the standard example.

Recall that the differential of a function \(f\) of \(x^1, \ldots, x^n\) corresponding to increments \(dx^i\) in the \(x^i\) is

\[
df = \frac{\partial f}{\partial x^i} dx^i
\]

Since \(dx^i\) is a vector \(df\) has the same value in any coordinate system.

In general, given any co-vector \((T_i)\), the differential form \(T_i dx^i\) is invariant under a change of a chart.

We identify \(dx^i \equiv e^i\) with the canonical basis of co-vectors or cotangent space.
5.2  Tensors of type $(0, 2)$

A basis for the space of tensors of type $(0, 2)$ at a given point are the products

$$e^i \otimes e^j$$

In terms of this basis an arbitrary tensor $T_{ij}$ has the form

$$T_{ij} e^i \otimes e^j$$

and can be regarded as a bilinear form on vectors since if $\xi$, $\eta$ are vectors then the scalar

$$T_{ij} \xi^i \eta^j$$

can be considered as the value of the bilinear form on those vectors.
Any $T_{ij}$ can be expressed as

$$T_{ij} = T_{ij}^{\text{sym}} + T_{ij}^{\text{alt}}$$

$$T_{ij}^{\text{sym}} = \frac{1}{2}(T_{ij} + T_{ji}) = T_{ji}^{\text{sym}}$$

$$T_{ij}^{\text{alt}} = \frac{1}{2}(T_{ij} - T_{ji}) = -T_{ji}^{\text{alt}}$$

A basis of $T_{ij}^{\text{sym}}$ is

$$\frac{e^i \otimes e^j + e^j \otimes e^i}{2}, \quad i \leq j$$

A basis of $T_{ij}^{\text{alt}}$ is

$$e^i \otimes e^j - e^j \otimes e^i, \quad i < j$$

Then

$$T_{ij}^{\text{sym}} e^i \otimes e^j = T_{ij}^{\text{sym}} \frac{e^i \otimes e^j + e^j \otimes e^i}{2}$$

$$= \sum_i T_{ii}^{\text{sym}} e^i \otimes e^i + \sum_{i<j} 2T_{ij}^{\text{sym}} \frac{e^i \otimes e^j + e^j \otimes e^i}{2}$$

and

$$T_{ij}^{\text{alt}} e^i \otimes e^j = T_{ij}^{\text{alt}} \frac{e^i \otimes e^j - e^j \otimes e^i}{2}$$

$$= \sum_{i<j} T_{ij}^{\text{alt}} (e^i \otimes e^j - e^j \otimes e^i)$$

In differential notation we identify

$$\frac{e^i \otimes e^j + e^j \otimes e^i}{2} \equiv dx^i dx^j = dx^j dx^i$$

$$e^i \otimes e^j - e^j \otimes e^i \equiv dx^i \wedge dx^j = -dx^j \wedge dx^i$$
5.3 Skew-symmetric Tensors of type $(0, k)$

**Definition.** A *skew-symmetric* tensor of type $(0, k)$ is a tensor $T_{i_1\cdots i_k}$ satisfying

$$T_{\sigma(i_1\cdots i_k)} = s(\sigma)T_{i_1\cdots i_k}$$

where for all permutations $\sigma$

$$s(\sigma) = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \end{cases}$$

Thus, if two indices are equal then the corresponding component of $T_{i_1\cdots i_k}$ is equal to 0.

Then, if $k > n$ the tensor is identically 0.

In what follows we assume $k \leq n$. 
The standard basis at a given point is

\[ dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k \]

where

\[ dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathbf{s}(\sigma) e^{\sigma(i_1)} \otimes \cdots \otimes e^{\sigma(i_k)} \]

Here \( S_k \) is the symmetric group, i.e. the group of all permutations of \( 1, \ldots, k \), and \( \sigma(l) \equiv i_{\sigma(l)} \).

The differential form of the skew-symmetric tensor \( (T_{i_1 \cdots i_k}) \) is

\[
T_{i_1 \cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = \sum_{i_1 < \cdots < i_k} T_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{k!} T_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\]
Example 1. A skew-symmetric tensor $T_{i_1\ldots i_n}$ of type $(0, n)$ in $n$-dim manifold is determined by the single component $T_{1\ldots n}$

$$T_{\sigma(1\ldots n)} = s(\sigma)T_{1\ldots n}$$

Thus, the space of skew-symmetric tensors of type $(0, n)$ is one-dim.

We denote $T_{i_1\ldots i_n}$ with $T_{1\ldots n} = 1$ as

$$\epsilon_{i_1\ldots i_n}$$

It is called the **Levi-Civita** symbol (or tensor) of rank $n$.

**Theorem.** Skew-symmetric tensors of type $(0, n)$ where $n$ is the dimension of the manifold $M$ transform as

$$(p)T_{1\ldots n} = (q)T_{1\ldots n} \cdot J$$

where $J$ is the Jacobian

$$J = \det \left( \frac{\partial x^i_q}{\partial x^j_p} \right)$$

HW: Prove the theorem
**Example 2.**

Let $G = (g_{ij})$ be a non-degenerate tensor, i.e. $g \equiv \det(g_{ij}) \neq 0$ (it does *not* have to be symmetric).

Then

$$g^{(p)}_{ij} = \frac{\partial x^k_q}{\partial x^l_p} \frac{\partial x^l_q}{\partial x^j_p} g^{(q)}_{kl} = \frac{\partial x^k_q}{\partial x^i_p} \frac{\partial x^l_q}{\partial x^j_p} g^{(q)}_{kl} \frac{\partial x^j_q}{\partial x^p_l}$$

or in matrix notations

$$g^{(p)}_{ij} = (A^T G^{(q)} A)_{ij} , \quad A = (A^l_j) = \left(\frac{\partial x^l_q}{\partial x^i_p}\right) \implies g^{(p)} = (\det A)^2 g^{(q)}$$

Thus, if $\det A > 0$ then

$$\sqrt{|g^{(p)}|} = \sqrt{|g^{(q)}|} \det A = \sqrt{|g^{(q)}|} J$$

Comparing with

$$(p)T_{1...n} = (q)T_{1...n} \cdot J$$

one concludes

The expression $\sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^n$ behaves as a tensor under coordinate changes for which the Jacobian $J = \det \left(\frac{\partial x^i_q}{\partial x^p_l}\right)$ is positive.

Since $J > 0$, the manifold $M$ is oriented.
Metric and distance function.

A metric \( g_{ij} \) on a manifold is a tensor of type \((0, 2)\), and on an oriented manifold such a metric gives rise to a \textit{volume element}

\[
T_{i_1 \ldots i_k} = \sqrt{|g|} \epsilon_{i_1 \ldots i_k}, \quad g = \det(g_{ij})
\]

It is convenient to write the volume element in the notation of differential forms

\[
\Omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n
\]

If \( g_{ij} \) is Riemann then the \textit{volume} \( V \) of \( M \) is

\[
V = \int_M \Omega = \int_M \sqrt{g} dx^1 \wedge \cdots \wedge dx^n
\]

A Riemann metric \( ds^2 = g_{ij}dx^i dx^j \) on a connected manifold \( M \) gives rise to a metric space structure on \( M \) with \textit{distance function} \( \rho(x, y) \) defined by

\[
\rho(x, y) = \inf_{\gamma} \int_{\gamma} ds
\]

where the infimum is taken over all piece-wise smooth arcs joining the points \( x \) and \( y \).

The topology on \( M \) defined by this metric-space structure coincides with the Euclidean topology on \( M \).

6 The behaviour of Tensors under Mappings (vol 1, section 22)

6.1 Tensors of type \((k, 0)\)

Recall that a smooth map \( f \) from an \( m \)-dim manifold \( M \) to an \( n \)-dim manifold \( N \) gives rise \( \forall x \in M \) to the push-forward map of tangent spaces

\[
f_* : T_x M \mapsto T_{f(x)} N
\]
which in terms of coordinates $x^i$ in $U \subset M$, $x \in U$, and $y^a$ in $V \subset N$, $y \in U$ is written as

$$y^a = f^a(x^1, \ldots, x^m), \quad a = 1, \ldots, n$$

\[ f_* : \xi^i \mapsto \eta^a = \frac{\partial f^a}{\partial x^i} \xi^i \]

This can be generalised to a push-forward map of the spaces of tensors of type $(k, 0)$

\[ f_* : \xi^{i_1 \cdots i_k} \mapsto \eta^{a_1 \cdots a_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi^{i_1 \cdots i_k} \]
6.2 Tensors of type $(0, k)$

Let $T_{x}^{(0,k)} M$ denote the space of tensors of type $(0, k)$ at $x \in M$.
Let $f$ be a smooth map $f$ from an $m$-dim manifold $M$ to an $n$-dim manifold $N$. It gives rise to a map

$$f^*: T_{f(x)}^{(0,k)} N \hookrightarrow T_{x}^{(0,k)} M$$

which in terms of coordinates $x^i$ in $U \subset M$, $x \in U$, and $y^a$ in $V \subset N$, $y \in U$ is written as

$$f^*: \eta_{a_1 \cdots a_k} \mapsto \xi_{i_1 \cdots i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1 \cdots a_k}$$

The map $f^*$ is called the pullback.

Let us denote

$$\zeta(\theta) \equiv \zeta_{i_1 \cdots i_k} \theta^{i_1 \cdots i_k}$$

the full contraction of a tensor of type $(0, k)$ with a tensor of type $(k, 0)$.

If $\xi_{i_1 \cdots i_k}$ and $\tilde{\eta}_{a_1 \cdots a_k}$ are tensors of type $(k, 0)$ and $(0, k)$ in $T_{x}^{(k,0)} M$ and $T_{y}^{(0,k)} N$, respectively, then

$$(f^* \tilde{\eta})(\xi) = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \tilde{\eta}_{a_1 \cdots a_k} \xi_{i_1 \cdots i_k}$$

$$= \tilde{\eta}_{a_1 \cdots a_k} \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi_{i_1 \cdots i_k} = \tilde{\eta}(f^* \xi)$$
Embeddings and Immersions of Manifolds

Definition 1.3.1a. A manifold $M$ of dim $m$ is said to be immersed in a manifold $N$ of dim $n \geq m$ if $\exists$ a smooth map $f : M \hookrightarrow N$ such that the push-forward map $f_*$ is at each point a one-to-one map of the tangent space. The map $f$ is called an immersion of $M$ into $N$.

Since $f_*$ is at each point a one-to-one map of the tangent space, in terms of local coordinates the Jacobian matrix of $f$ at each point has rank equal to $m = \dim M$.

Definition 1.3.1b. An immersion of $M$ into $N$ is called embedding if it is one-to-one. Then, $M$ is called a submanifold of $N$. 
**Example 1.** Suppose $M$ is an $m$-dim submanifold of an $n$-dim (pseudo-)Riemann manifold $N$ with metric $g_{ab}^{(N)}$. Then, the pullback of $g_{ab}^{(N)}$ to $M$ yields the tensor

$$g_{ij}^{(M)}(x) = \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} g_{ab}^{(N)}(f(x)), \quad i, j = 1, \ldots, m, \quad a, b = 1, \ldots, n$$

which is called the metric *induced* on the submanifold $M$ by the metric $g_{ab}^{(N)}$ of $N$.

Consider the line element $ds_N$ of $N$

$$ds_N^2 = g_{ab}^{(N)}(y) dy^a dy^b$$

Let

$$y^a = f^a(x^1, \ldots, x^m), \quad a = 1, \ldots, n$$

Then, the line element becomes

$$ds_N^2 = g_{ab}^{(N)}(f(x)) \frac{\partial f^a}{\partial x^i} dx^i \frac{\partial f^b}{\partial x^j} dx^j = g_{ij}^{(M)}(x) dx^i dx^j = ds_M^2$$

Thus, the infinitesimal distances measured by using the metric on $N$ and the metric induced on $M$ are the same.

A simple example is $S^2$ in $\mathbb{R}^3$. 
Example 2. Consider the pullback of a skew-symmetric tensor $T_{a_1\ldots a_m}$ of type $(0,m)$, i.e. an $m$-form, to an $m$-dim submanifold $M$, $y^a = f^a(x^1, \ldots, x^m)$, $a = 1, \ldots, n$ in an $n$-dim manifold $N$.

Theorem 22.1.2. The pullback of the skew-symmetric form

$$\frac{1}{m!} T_{a_1\ldots a_m} dy^{a_1} \wedge \cdots \wedge dy^{a_m}$$

to the $m$-dim submanifold $y^a = f^a(x^1, \ldots, x^m)$ is given by

$$\left( \frac{1}{m!} J^{a_1\ldots a_m} T_{a_1\ldots a_m} \right) dx^1 \wedge \cdots \wedge dx^m$$

where $J^{a_1\ldots a_m}$ is the $m \times m$ minor of the matrix $(\partial y^a / \partial x^i)$ formed from the columns numbered $a_1, \ldots, a_m$.

Thus, we have on $M$

$$\frac{1}{m!} T_{a_1\ldots a_m} dy^{a_1} \wedge \cdots \wedge dy^{a_m} = \left( \frac{1}{m!} J^{a_1\ldots a_m} T_{a_1\ldots a_m} \right) dx^1 \wedge \cdots \wedge dx^m$$

Proof. By definition of the pullback

$$\tilde{T}_{1\ldots m} = \frac{\partial y^{a_1}}{\partial x^1} \cdots \frac{\partial y^{a_m}}{\partial x^m} T_{a_1\ldots a_m}$$

$$= \sum_{a_1 < \cdots < a_m} T_{a_1\ldots a_m} \left( \sum_{\sigma \in S_m} \mathfrak{s}(\sigma) \frac{\partial y^{a_{\sigma(1)}}}{\partial x^1} \cdots \frac{\partial y^{a_{\sigma(m)}}}{\partial x^m} \right)$$

$$= \sum_{a_1 < \cdots < a_m} T_{a_1\ldots a_m} J^{a_1\ldots a_m} = \frac{1}{m!} J^{a_1\ldots a_m} T_{a_1\ldots a_m}$$
7 Embeddings and Immersions of Manifolds

**Definition 1.3.1a.** A manifold $M$ of dim $m$ is said to be immersed in a manifold $N$ of dim $n \geq m$ if $\exists$ a smooth map $f : M \hookrightarrow N$ such that the push-forward map $f_*$ is at each point a one-to-one map of the tangent space. The map $f$ is called an immersion of $M$ into $N$.

Since $f_*$ is at each point a one-to-one map of the tangent space, in terms of local coordinates the Jacobian matrix of $f$ at each point has rank equal to $m = \dim M$.

**Definition 1.3.1b.** An immersion of $M$ into $N$ is called embedding if it is one-to-one. Then, $M$ is called a submanifold of $N$. 
We always assume that any submanifold $M$ is defined in each chart $U_p$ of the containing manifold $N$ by a system of eqs

$$\begin{aligned}
&f^1_p(x^1_p,\ldots,x^n_p) = 0 \\
f^2_p(x^1_p,\ldots,x^n_p) = 0 \\
&\quad \vdots \\
f^{n-m}_p(x^1_p,\ldots,x^n_p) = 0
\end{aligned}$$

where $\text{rank} \left( \frac{\partial f^i_p}{\partial x^\alpha_p} \right) = n - m$

with the property that on each intersection $U_p \cap U_q$ the systems $(f^i_p = 0)$ and $(f^i_q = 0)$ have the same set of zeroes.

Let us introduce in each $U_p \subset N$ new coordinates $y^1_p,\ldots,y^n_p$ satisfying

$$y^{m+1}_p = f^1_p(x^1_p,\ldots,x^n_p), \quad y^{m+2}_p = f^2_p(x^1_p,\ldots,x^n_p), \quad \ldots, \quad y^n_p = f^{n-m}_p(x^1_p,\ldots,x^n_p)$$

Then, $M$ is given by

$$y^{m+1}_p = 0, \quad y^{m+2}_p = 0, \quad \ldots, \quad y^n_p = 0$$

while $y^1_p,\ldots,y^m_p$ serve as local coordinates on $M$. 


Definition 1.3.2. A closed region $A$ of a manifold $M$ defined by an inequality $f(x) \leq 0$ (or $f(x) \geq 0$) where $f$ is a real-valued function on $M$ is called a manifold with boundary.

It is assumed that the boundary $\partial A$ given by $f(x) = 0$ is a non-singular submanifold of $M$, i.e. $\vec{\nabla} f \neq 0$ on $\partial A$.

Let $A$ and $B$ be manifolds with boundary, both given as closed regions of manifolds $M$ and $N$. A map

$$\varphi : A \mapsto B$$

is said to be a smooth map of manifolds with boundary if it is a restriction to $A$ of a smooth map

$$\tilde{\varphi} : U \mapsto N, \quad \tilde{\varphi}|A = \varphi$$

of an open region $U$ of $M$ containing $A$, e.g. if

$$A : f(x) \leq 0 \text{ then } U \text{ is } U_\varepsilon = \{x | f(x) < \varepsilon\}, \quad \varepsilon > 0.$$

Definition. A compact manifold without boundary is called closed.
8 Surfaces in Euclidean space

8.1 Surfaces as Manifolds

Definition. A non-singular surface $M$ of dimension $k$ in $n$-dim Euclidean space is given by a set of $n - k$ eqs

$$f_i(x^1, \ldots, x^n) = 0, \quad i = 1, \ldots, n - k$$

(8.4)

where $\forall x$ the matrix $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ has rank $n - k$.

Let $J_{j_1 \cdots j_{n-k}}$ be the minor of a submatrix made up of the columns of $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$ which are indexed by $j_1, \ldots, j_{n-k}$.

Let $U_{j_1 \cdots j_{n-k}}$ be the region consisting of all points of the surface at which $J_{j_1 \cdots j_{n-k}}$ does not vanish.

Obviously,

$$M = \bigcup_{j_1 \cdots j_{n-k}} U_{j_1 \cdots j_{n-k}}$$

Since $J_{j_1 \cdots j_{n-k}} \neq 0$ on $U_{j_1 \cdots j_{n-k}}$, we can take

$$(y^1, \ldots, y^k) = (x^1, \ldots, \hat{x}^{j_1}, \ldots, \hat{x}^{j_{n-k}}, \ldots, x^n)$$

(8.5)

as local coordinates on $U_{j_1 \cdots j_{n-k}}$.

Theorem 2.1.1. The covering of the surface $M$ (8.4) by the regions

$$U_{j_1 \cdots j_{n-k}}, \quad 1 \leq j_1 < \cdots < j_{n-k} \leq n$$

each furnished with local coordinates (8.5), defines on the surface the structure of a smooth manifold.

Proof. Blackboard
**Remark 1.** The Jacobian of the transition function $y \to z$ is

$$J_{(y)\to(z)} = \pm \frac{J_{s_1 \cdots s_{n-k}}}{J_{j_1 \cdots j_{n-k}}}$$

HW: Prove it.

**Remark 2.** The tangent space to the surface $M$ (8.4) is identifiable with the linear subspace of $\mathbb{R}^n$ consisting of the solutions of the system

$$\frac{\partial f_1}{\partial x^\alpha} \xi^\alpha = 0, \ldots, \frac{\partial f_{n-k}}{\partial x^\alpha} \xi^\alpha = 0$$

Thus, the co-vectors $\vec{\nabla} f_i = (\frac{\partial f_1}{\partial x^\alpha}), i = 1, \ldots, n - k$ are orthogonal to the surface at each point.
8.2 Surfaces can be oriented

Consider at any point $x$ of an $n$-dim manifold $M$ the various frames (i.e. ordered bases)

$$\tau = (e_1, \ldots, e_n)$$

for the tangent space to $M$ at $x$. Any two such frames

$$\tau_1 = (e^{(1)}_1, \ldots, e^{(1)}_n) \quad \text{and} \quad \tau_2 = (e^{(2)}_1, \ldots, e^{(2)}_n)$$

are related by a nonsingular linear transformation $A$

$$A : e^{(1)}_k \rightarrow e^{(2)}_k, \quad k = 1, \ldots, n$$

**Definition.** We say the ordered bases $\tau_1, \tau_2$ *lie in the same orientation class* if $\det A > 0$, and *lie in opposite orientation classes* if $\det A < 0$.

**Definition 2.1.2.** A manifold is said to be *orientable* if it is possible to choose at every point of it a single orientation class depending continuously on the points.

A particular choice of such an orientation class for each point is called an *orientation* of the manifold, and a manifold equipped with a particular orientation is said to be *oriented*.

If no orientation exists the manifold is *non-orientable*. 
Theorem 2.1.3. Definition 1.1.3 is equivalent to Definition 2.1.2.

Proof of Def 1.1.3 ⇒ Def 2.1.2.
Let $M$ be oriented in the sense of Def 1.1.3. Then, we choose at each
\[ x \in U_j \subset M \]
as our orienting frame the $n$-tuple
\[ (e_{1j}, \ldots, e_{nj}) \]
consisting of the standard basis vectors tangent to the coordinate axes of the local coordinate system
\[ x^1_j, \ldots x^n_j \]
If
\[ x \in U_j \quad \text{and} \quad x \in U_k \]
then the two orienting frames are related by the Jacobian matrix of transition function. Since the Jacobian is positive the two frames lie in the same orientation class.

Proof of Def 1.1.3 ⇒ Def 2.1.2. See the textbook
Theorem 2.1.4. A smooth non-singular surface $M^k$ in $n$-dim space $\mathbb{R}^n$, defined by a system of eqs (8.4), is orientable.

Proof. Let $\tau$ denote a point-dependent tangent frame to the surface $M^k$. The ordered $n$-tuple

$$\hat{\tau} = (\tau, \vec{\nabla} f_1, \ldots, \vec{\nabla} f_{n-k})$$

of vectors is linearly independent at each point because $\vec{\nabla} f_i$ are linearly independent among themselves and orthogonal to the surface.

We can choose $\tau$ at each $x \in M^k$ so that $\hat{\tau}$ lies in the same orientation class as the standard frame

$$(e_1, \ldots, e_n)$$

Since this orientation class depends continuously on $x \in \mathbb{R}^n$, so will the orientation class of $\tau$ depend continuously on $x \in M^k$. 

Example 1. An $n$-sphere $S^n$ in $\mathbb{R}^{n+1}$: $x_1^2 + \cdots + x_{n+1}^2 = 1$.

The $n$-sphere bounds a manifold with boundary, denoted by $D^{n+1}$ and called the closed $(n + 1)$-dim disc or ball, defined by

$$f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1 \leq 0$$

Then $S^n$ separates $\mathbb{R}^{n+1}$ into two non-intersecting regions defined by $f(x) < 0$ and $f(x) > 0$.

Example 2. A hyperbolic $n$-space, $H^n$, is one sheet of the hyperboloid of two sheets realised as a surface

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = -1, \quad x_0 > 0,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 3. An $n$-dimensional de Sitter space, $dS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = 1,$$

in the Minkowski space $\mathbb{R}^{1,n}$

Example 4. An $n$-dimensional anti-de Sitter space, $AdS_n$ is the hyperboloid of one sheet realised as a surface

$$-x_{-1}^2 - x_0^2 + \sum_{i=1}^{n-1} x_i^2 = -1,$$

in the pseudo-Euclidean space $\mathbb{R}^{2,n-1}$
Definition 2.1.5. A connected $(n - 1)$-dim submanifold of $\mathbb{R}^n$ is called \textit{two-sided} if a single-valued continuous field of unit normals can be defined on it. Such a submanifold is called a two-sided hypersurface.

Theorem 2.1.6. A two-sided hypersurface in $\mathbb{R}^n$ is orientable.

Proof. See the textbook

It can be shown that any two-sided hypersurface in $\mathbb{R}^n$ is defined by a single non-singular eq $f(x) = 0$. Thus, it bounds a manifold with boundary.

Then, one can prove that any closed hypersurface in $\mathbb{R}^n$ is two-sided.
8.3 Transformation Groups as Surfaces

Definition. A group is a nonempty set $G$ on which there is defined a binary operation $(a, b) \mapsto ab$ satisfying the following properties

- **Closure:** If $a$ and $b$ belong to $G$, then $ab$ is also in $G$.
- **Associativity:** $a(bc) = (ab)c$ for all $a, b, c \in G$.
- **Identity:** There is an element $1 \in G$ such that $a1 = 1a = a$ for all $a$ in $G$.
- **Inverse:** If $a \in G$, then there is an element $a^{-1} \in G$: $aa^{-1} = a^{-1}a = 1$.

Examples of groups which are manifolds are

1. The *general* linear group $GL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with non-zero determinant is a region in $\mathbb{R}^{n^2}$

2. The *special* linear group $SL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with determinant equal to 1
   
   $$\det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a hypersurface in $\mathbb{R}^{n^2}$

3. The *orthogonal* group $O(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying

   $$A^T \cdot A = I, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a surface in $\mathbb{R}^{n^2}$

4. The *special orthogonal* group $SO(n, \mathbb{R})$ consisting of all $n \times n$ real matrices satisfying

   $$A^T \cdot A = I, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{R})$$

   is a surface in $\mathbb{R}^{n^2}$
5. The \textit{pseudo-orthogonal} group $O(p,q,\mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices satisfying
\[ A^T \cdot \eta \cdot A = \eta, \quad \eta = \text{diag} \left( 1, \ldots, 1, -1, \ldots, -1 \right), \quad A \in \text{Mat}(n, \mathbb{R}) \]
is a surface in $\mathbb{R}^{n^2}$

6. The \textit{special pseudo-orthogonal} group $SO(p,q,\mathbb{R})$ consisting of all $n \times n$, $n = p + q$ real matrices $A \in \text{Mat}(n, \mathbb{R})$ satisfying
\[ A^T \cdot \eta \cdot A = \eta, \quad \det A = 1, \quad \eta = \text{diag} \left( 1, \ldots, 1, -1, \ldots, -1 \right) \]
is a surface in $\mathbb{R}^{n^2}$

7. The \textit{unitary} group $U(n)$ consisting of all $n \times n$ complex matrices satisfying
\[ A^\dagger \cdot A = \mathbb{I}, \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$

8. E.g. $U(1) \simeq S^1$

9. The \textit{special unitary} group $SU(n)$ consisting of all $n \times n$ complex matrices satisfying
\[ A^\dagger \cdot A = \mathbb{I}, \quad \det A = 1, \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$

10. E.g. $SU(2) \simeq S^3$

11. The \textit{pseudo-unitary} group $U(p,q)$ consisting of all $n \times n$, $n = p + q$ complex matrices satisfying
\[ A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag} \left( 1, \ldots, 1, -1, \ldots, -1 \right), \quad A \in \text{Mat}(n, \mathbb{C}) \]
is a surface in $\mathbb{R}^{2n^2}$
12. The special pseudo-unitary group $SU(p, q)$ consisting of all $n \times n$, $n = p + q$ complex matrices $A \in \text{Mat}(n, \mathbb{C})$ satisfying

$$A^\dagger \cdot \eta \cdot A = \eta, \quad \eta = \text{diag} \left( 1, \ldots, 1, -1, \ldots, -1 \right), \quad \det A = 1$$

is a surface in $\mathbb{R}^{2n^2}$

**Definition 2.1.7.** A manifold $G$ is called a *Lie group* if it has given on it a group operation with the property that the maps $\varphi : G \mapsto G$, defined by $\varphi(g) = g^{-1}$ (i.e. the taking of inverses) and $\psi : G \times G \mapsto G$ defined by $\psi(g, h) = gh$ (i.e. the group multiplication), are smooth maps.
9 Projective Spaces

9.1 Real Projective Space

**Definition.** The *real projective space* $\mathbb{R}P^n$ is the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin. Equivalently it is the set of equivalence classes of nonzero vectors in $\mathbb{R}^{n+1}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.

Since each line passing through the origin intersect a sphere $S^n$ centred at the origin in exactly two points, the points of $\mathbb{R}P^n$ are in one-to-one correspondence with the pairs of diametrically opposite points of the $n$-sphere. We may think of $\mathbb{R}P^n$ as obtained from $S^n$ by gluing, that is identifying, diametrically opposite points. So, $\mathbb{R}P^n \cong S^n/Z_2$, where $Z_2$ maps a point of $S^n$ to the diametrically opposite point.

The projective line

$$\mathbb{R}P^1 \cong S^1/Z_2 \cong S^1 \cong U(1)$$

$\mathbb{R}P^2$ is called the projective plane
9.2 Quaternions, $SU(2)$, $SO(3)$, $\mathbb{R}P^3$

Definition. The set $\mathbb{H}$ of *quaternions* consists of all linear combinations

$$q \in \mathbb{H}, \quad q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R},$$

and $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are linearly independent (so it is a 4-dim vector space).

We introduce the following multiplication in $\mathbb{H}$

$$\begin{align*}
\mathbf{i} \cdot \mathbf{j} &= \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{i} = -\mathbf{k} \cdot \mathbf{j}, \quad \mathbf{k} \cdot \mathbf{i} = \mathbf{j} = -\mathbf{i} \cdot \mathbf{k}, \\
\mathbf{i} \cdot \mathbf{i} &= \mathbf{i}^2 = -\mathbf{1}, \quad \mathbf{j} \cdot \mathbf{j} = \mathbf{j}^2 = -\mathbf{1}, \quad \mathbf{k} \cdot \mathbf{k} = \mathbf{k}^2 = -\mathbf{1}, \\
\mathbf{i} \cdot \mathbf{1} &= \mathbf{i} = \mathbf{1} \cdot \mathbf{i}, \quad \mathbf{j} \cdot \mathbf{1} = \mathbf{j} = \mathbf{1} \cdot \mathbf{j}, \quad \mathbf{k} \cdot \mathbf{1} = \mathbf{k} = \mathbf{1} \cdot \mathbf{k}, \quad \mathbf{1} \cdot \mathbf{1} = \mathbf{1},
\end{align*}$$

which makes $\mathbb{H}$ an associative algebra over the field of real numbers.
For each quaternion
\[ q = a \mathbf{1} + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}, \quad a, b, c, d \in \mathbb{R}, \]
we define
\[ A(q) = \begin{pmatrix}
    a + b i & c + d i \\
    -c + d i & a - b i
\end{pmatrix}, \quad A(q) \in \text{Mat}(2, \mathbb{C}). \]

**Lemma 1.** The map \( q \mapsto A(q) \) is one-to-one and
\[ A(q_1q_2) = A(q_1)A(q_2) \]
so that this map is an algebra *monomorphism* which means it is an injective (one-to-one) homomorphism (consistent with multiplication).

Note
\[ A(\mathbf{i}) = i\sigma_3, \quad A(\mathbf{j}) = i\sigma_2, \quad A(\mathbf{k}) = i\sigma_1, \]
where \( \sigma_i \) are the Pauli matrices.
We define an operation of *conjugation* on $\mathbb{H}$ by
\[ \bar{q} = a \mathbf{1} - b \mathbf{i} - c \mathbf{j} - d \mathbf{k}, \quad a, b, c, d \in \mathbb{R} \]

**Lemma 2.** The map $q \mapsto \bar{q}$ is an anti-isomorphism of $\mathbb{H}$, i.e. it is linear, and
\[ q_1 q_2 = \bar{q}_2 \bar{q}_1 \]

We define the *norm* $|q| \geq 0$ of a quaternion by
\[ |q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2, \quad a, b, c, d \in \mathbb{R} \]

Then, if $|q| \neq 0$ then
\[ q \cdot \frac{\bar{q}}{|q|^2} = \mathbf{1} \]
and therefore each nonzero quaternion has multiplicative inverse
\[ q^{-1} = \frac{\bar{q}}{|q|^2} \]

**Lemma 3.** The algebra $\mathbb{H}$ of quaternions is a *division* algebra, i.e. for each nonzero quaternion $q$ there exists a quaternion $q^{-1}$ inverse to it
\[ qq^{-1} = \mathbf{1} = q^{-1}q \]
We have
\[ |q|^2 = \det A(q) \implies |q_1 q_2|^2 = |q_1|^2 |q_2|^2 \]
Thus, the set of quaternions with norm 1 forms a group under multiplication. It is denoted by $\mathbb{H}_1$.

For $q \in \mathbb{H}_1$ we have $q^{-1} = \bar{q}$.

If we regard $\mathbb{H}$ as $\mathbb{R}^4$ then $\mathbb{H}_1$ is the hypersurface
\[ |q|^2 = a^2 + b^2 + c^2 + d^2 = 1 \implies \mathbb{H}_1 \cong S^3 \cong SU(2) \]
Let $\mathbb{H}_0$ denote the 3-dim Euclidean space consisting of all quaternions $x$ satisfying $\bar{x} = -x$, i.e. with the real part zero. Then

$$|x|^2 = x \cdot \bar{x} = -x^2 \geq 0$$

**Lemma 4.** If $|q|^2 = 1$, then the transformation defined by

$$\alpha_q : x \mapsto qxq^{-1}, \quad x \in \mathbb{H}_0$$

is a rotation of 3-dim Euclidean space $\mathbb{H}_0 = \mathbb{R}^3$.

Thus, the map $q \mapsto \alpha_q$ is a homomorphism from $\mathbb{H}_1 \cong SU(2)$ to the group of rotations of $\mathbb{R}^3$.

Obviously, $\alpha_q$ is the identity map when $q = \pm 1$.

Check that every rotation is of the form (9.6).

Thus, $SO(3)$ is isomorphic to $SU(2)/\{1,-1\}$.

Since $\{1,-1\} \cong \mathbb{Z}_2$, where $\mathbb{Z}_2$ identifies $A$ and $-A$ which correspond to diametrically opposite points of $S^3 \cong SU(2)$, we get

$$SO(3) \cong SU(2)/\mathbb{Z}_2 \cong S^3/\mathbb{Z}_2 \cong \mathbb{R}P^3$$

**Remark.** $\mathbb{H}_1 \cong SU(2)$ can be also used to prove that

$$SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$$
### 9.3 \( \mathbb{R}P^n \) as a manifold

Let’s introduce a manifold structure on \( \mathbb{R}P^n \).

\( \mathbb{R}P^n \) as the set of equivalence classes of nonzero vectors in \( \mathbb{R}^{n+1} \) with coordinates \( y^0, \ldots, y^n \).

For each \( q = 0, 1, \ldots, n \), let \( U_q \) denote the set of equivalence classes of vectors \( (y^0, \ldots, y^n) \) with \( y^q \neq 0 \). Obviously, \( \mathbb{R}P^n = \bigcup_{q=0}^{n} U_q \).

We introduce the following local coordinates on \( U_0 \)

\[
x^i_0 = \frac{y^i}{y^0}, \quad i = 1, \ldots, n.
\] (9.7)

Then, the local coordinates on each \( U_q \) are introduced through the following recursion relations

\[
x^{i+1}_q = \frac{x^i_q}{x^1_q}, \quad i = 1, \ldots, n - 1, \quad x^n_{q+1} = -\frac{1}{x^1_q}.
\] (9.8)

One gets

\[
x^1_q = \frac{x^2_{q-1}}{x^1_{q-1}} = \frac{x^3_{q-2} x^1_{q-2}}{x^2_{q-2} x^2_{q-2}} = \frac{x^3_{q-2}}{x^2_{q-2}} = \frac{x^4_{q-3} x^1_{q-3}}{x^3_{q-3} x^3_{q-3}} = \frac{x^4_{q-3}}{x^3_{q-3}} = \ldots
\] (9.9)

and therefore \( x^1_q \neq 0 \) on \( U_{q+1} \) and up to a sign \( x^i_{q+1} \) can be expressed as the ratio \( y^{n_i}/y^{q+1} \) where the index \( n_i \) depends on \( i \) and \( q \). Eqs (9.8) also provide transition functions on the intersection \( U_q \cap U_{q+1} \).
\begin{equation}
    x_{q+1}^i = \frac{x_{q}^{i+1}}{x_1^q}, \quad i = 1, \ldots, n - 1, \quad x_{q+1}^n = -\frac{1}{x_1^q}.
\end{equation} \tag{9.10}

\begin{equation}
    x_1^q = \frac{y^{q+1}}{y^q}.
\end{equation} \tag{9.11}

The Jacobian of these transition functions can be easily found

\begin{equation}
    J_{(x_q)\rightarrow(x_{q+1})} = \left(-\frac{1}{x_1^q}\right)^{n+1} \neq 0. \tag{9.12}
\end{equation}

Since, the Jacobian of the transition functions for arbitrary \( q \) and \( p \) is

\begin{equation}
    J_{(x_q)\rightarrow(x_p)} = J_{(x_q)\rightarrow(x_{q+1})}J_{(x_{q+1})\rightarrow(x_{q+2})}\cdots J_{(x_{p-1})\rightarrow(x_p)}
\end{equation}

\begin{equation}
    = \prod_{k=q}^{p-1} \left(-\frac{1}{x_k^1}\right)^{n+1} = \prod_{k=q}^{p-1} \left(-\frac{y^k}{y^{k+1}}\right)^{n+1}
\end{equation} \tag{9.13}

\begin{equation}
    = \left((-1)^{p-q} y^{q-p}\right)^{n+1} \neq 0,
\end{equation}

and it is a smooth function on \( U_q \cap U_p \), we conclude that the real projective space \( \mathbb{R}P^n \) is a smooth manifold.

Moreover, \( \mathbb{R}P^n \) is oriented for odd \( n \).

\( \mathbb{R}P^2 \) is called the \textit{projective plane}, and \( U_0 \) is called the finite part of the projective plane.
9.4 Complex Projective Space

**Definition.** The *complex projective space* \( \mathbb{C}P^n \) is the set of equivalence classes of nonzero vectors in \( \mathbb{C}^{n+1} \) where two nonzero vectors are equivalent if they are scalar multiples of one another.

The charts are defined as in the real case, making \( \mathbb{C}P^n \) a \( 2n \)-dim smooth manifold.

Consider the *complex projective line* \( \mathbb{C}P^1 \). Its points are equivalence classes of nonzero pairs

\[
(z^0, z^1) \sim (\lambda z^0, \lambda z^1), \quad \lambda \neq 0, \lambda \in \mathbb{C}
\]

Consider the complex function

\[
w_0(z^0, z^1) = \frac{z^1}{z^0}
\]

defined on \( U_0 : z^0 \neq 0 \) which covers all \( \mathbb{C}P^1 \) except the equivalence class of \((0, 1)\) which contains all nonzero pairs of the form \((0, z^1)\).

We define \( w_0 \) as taking the value \( \infty \) at this point.

Then, via the function \( w_0 \), \( \mathbb{C}P^1 \) becomes identified with the *extended complex plane*, i.e. the union of the ordinary complex plane with a point at infinity.

**Theorem 2.2.1.** The complex projective line \( \mathbb{C}P^1 \) is diffeomorphic to the 2-dim sphere \( S^2 \).

**Proof.** Blackboard

By this result, the extended complex plane is often called the Riemann sphere.
The general complex projective space $\mathbb{C}P^n$

From each equivalence class of $(n+1)$-dim vectors $(z_0, z_1, \ldots, z^n)$ we choose as a representative a vector with the unit norm

$$|z_0|^2 + |z_1|^2 + \cdots |z^n|^2 = 1$$

This equation defines the unit sphere $S^{2n+1}$ in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.

This representative is not unique because one can multiply all $z^i$ by $e^{i\varphi}$, $\varphi \in \mathbb{R}$, i.e. by complex numbers of modulus 1 which form the group $U(1)$.

Thus,

The complex projective space $\mathbb{C}P^n$ can be obtained from the unit sphere $S^{2n+1} = \{ z \mid \sum_{i=0}^{n} |z|^2 = 1 \}$, by identifying all points differing by a scalar factor of the form $e^{i\varphi}$.

This identification provides a map

$$S^{2n+1} \mapsto \mathbb{C}P^n \cong S^{2n+1}/U(1)$$

such that the pre-image of each point of $\mathbb{C}P^n$ is topologically equivalent to the circle $S^1$.

In particular, since $\mathbb{C}P^1 \cong S^2$, we get a map

$$S^3 \mapsto S^2, \quad (z^0, z^1) \mapsto w = \frac{z^1}{z^0}, \quad |z^0|^2 + |z^1|^2 = 1$$

This map is called the Hopf bundle, or fibration or map. $S^1$ is the fibre space embedded in $S^3$, and the Hopf map projects $S^3$ onto the base space $S^2$. 
10 Elements of the Theory of Lie Groups

10.1 Neighbourhood of the Identity Element of a Lie Group

Let \( G \) be a Lie group, let the point \( g_0 = 1 \in G \) be the identity element of \( G \), and let \( T = T(1) \) be the tangent space at the identity element.

Let us express the group operations on \( G \) in a chart \( U_0 \) containing \( g_0 \) in terms of local coordinates.

We choose coordinates in \( U_0 \) so that the identity element is the origin
\[
1 = g_0 = (0, \ldots, 0)
\]

Let
\[
g_1 = (x^1, \ldots, x^n), \quad g_2 = (y^1, \ldots, y^n), \quad g_3 = (z^1, \ldots, z^n)
\]
where \( g_k \) are such that
\[
g_k, g_k^{-1}, g_k g_l, g_k^{-1} g_l^{-1}, g_k g_l^{-1}, g_k^{-1} g_l \in U_0, \quad k, l = 1, 2, 3
\]

Then,
\[
g_1 g_2 = (\psi^1(x,y), \psi^2(x,y), \ldots, \psi^n(x,y)) = (\psi^i(x,y)),
\]
\[
\psi^i(x,y) = \psi^i(x^1, \ldots, x^n, y^1, \ldots, y^n), \quad i = 1, \ldots, n
\]
\[
g_1^{-1} = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^n(x)) = (\varphi^i(x)),
\]
\[
\varphi^i(x) = \varphi^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n
\]
are the coordinates of \( g_1 g_2 \) and \( g_1^{-1} \).

\( \psi(x,y) \) and \( \varphi(x) \) satisfy ( \( i = 1, \ldots, n \) )
1. \( \psi^i(x, 0) = \psi^i(0, x) = x^i \)
2. \( \psi^i(x, \varphi(x)) = 0 \)
3. \( \psi^i(x, \psi(y, z)) = \psi^i(\psi(x, y), z) \)
Let $\psi(x, y)$ be sufficiently smooth. Then
\[
\psi^i(x, y) = x^i + y^i + b^i_{jk}x^j y^k + \text{(terms of order)} \geq 3
\]
\[
b^i_{jk} = \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \bigg|_{x=y=0}
\]

Let $\xi, \eta \in T$, and their components in terms of $x^i$ are $\xi^i$ and $\eta^i$.

**Definition.** The *commutator* $[\xi, \eta] \in T$ is defined by
\[
[\xi, \eta]^i = c^i_{jk} \xi^j \eta^k, \quad c^i_{jk} \equiv b^i_{jk} - b^i_{kj}
\]

It has 3 basic properties

1. $[\cdot, \cdot]$ is a bilinear operation on the $n$-dim vector space $T$
2. Skew-symmetry: $[\xi, \eta] = -[\eta, \xi]$
3. Jacoby’s identity: $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$

**Proof of 3.** Blackboard
Definition. A Lie algebra is a vector space $G$ over a field $F$ with a bilinear operation $[\cdot, \cdot]: G \times G \rightarrow G$ which is called a commutator or a Lie bracket, such that the following axioms are satisfied:

- It is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in G$
- It satisfies the Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Thus, the tangent space of a Lie group $G$ at the identity is with respect to the commutator operation a Lie algebra called the Lie algebra of the Lie group $G$.

Let $e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n}$ be the standard basis vectors of $T$ in terms of the coordinates $x^1, \ldots, x^n$.

Let us multiply $[\xi, \eta] = c^i_{jk} \xi^j \eta^k$ by $e_i$ and take the sum over $i$

$$[\xi, \eta] = c^i_{jk} \xi^j \eta^k e_i$$

Let us choose $\xi = e_j, \eta = e_k$. Taking into account that the components of the standard basis vector are

$$\left( e_m \right)^n = \delta^n_m$$

we get

$$[e_j, e_k] = c^i_{jk} e_i$$

Definition. The constants $c^i_{jk}$ which determine the commutation operation on a Lie algebra, and which are skew-symmetric in $j, k$, are called the structure constants of the Lie algebra.
10.2 One-parameter subgroups and canonical coordinates

**Definition.** A one-parameter subgroup of a Lie group $G$ is defined to be a parametric curve $F(t)$ on the manifold $G$ such that

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F(t)^{-1}$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = \frac{dF(t + \epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \frac{d}{d\epsilon} (F(t)F(\epsilon)) \bigg|_{\epsilon=0} = F(t) \frac{dF(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0}$$

Hence,

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F(t)^{-1}\dot{F}(t) = \dot{F}(0),$$

i.e. the induced action of left multiplication by $F(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0) = \text{const} \in T$.

Conversely, $\forall A \in T$, the equation

$$F(t)^{-1}\dot{F}(t) = A$$

is satisfied by a unique one-parameter subgroup $F(t)$ of $G$.

If $G$ is a matrix group then $F(t) = \exp At$.

We will use this notation for arbitrary Lie groups.
Let’s discuss how $F(t)$ and the push-forward map look like in a neighbourhood of $U_0$.

Let $F(t) \in U_0$ have local coordinates $f^1(t), \ldots, f^n(t)$. Since $F(t)$ is a one-parameter subgroup the functions $f^i(t)$ satisfy

$$f^i(0) = 0, \quad f^i(t_1 + t_2) = \psi^i(f(t_1), f(t_2)), \quad f^i(-t) = \varphi^i(f(t))$$

Consider the map $G \mapsto G$ given by the left multiplication by $F(t)$

$$x \mapsto y = F(t)x, \quad x, y \in G$$

If the local coordinates of $x$ and $y$ are $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ then the left multiplication takes the form

$$x \mapsto y : y^i = \psi^i(f(t), x)$$

The corresponding push-forward map induced by the left multiplication is

$$F_*(t) : \xi^i \mapsto \eta^i = \frac{\partial \psi^i(f(t), x)}{\partial x^j}\xi^j, \quad \xi \in T_xG, \quad \eta \in T_yG$$

The velocity vector at $F(t)$ is

$$\frac{dF}{dt} = (\dot{f}^1(t), \ldots, \dot{f}^n(t)), \quad \text{where}$$

$$\dot{f}^i(t) = \frac{df^i(t + \epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d\psi^i(f(t), f(\epsilon))}{d\epsilon}\bigg|_{\epsilon=0} = \frac{\partial \psi^i(f(t), x)}{\partial x^j}\bigg|_{x=f(t)} \cdot \dot{f}^j(0)$$

Thus, $F_*(t)$ sends $\dot{F}(0)$ to $\dot{F}(t)$. Similarly, we get

$$\dot{f}^i(0) = \frac{df^i(-t + \epsilon)}{d\epsilon}\bigg|_{\epsilon=t} = \frac{d\psi^i(f(-t), f(\epsilon))}{d\epsilon}\bigg|_{\epsilon=t} = \frac{\partial \psi^i(\varphi(f(t)), x)}{\partial x^j}\bigg|_{x=f(t)} \cdot \dot{f}^j(t)$$

and therefore $F_*(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0)$. 
**Definition.** For each $h \in G$ the transformation $G \mapsto G$ defined by $g \mapsto hgh^{-1}$ is called the *inner automorphism* of $G$ determined by $h$.

Any inner automorphism does not move $g_0$, $hg_0h^{-1} = g_0$, and therefore the push-forward (induced linear) map of the tangent space $T$ to $G$ at $g_0$ is a linear transformation of $T$ denoted by (or by Adj)

$$\text{Ad} (h) : T \mapsto T$$

It satisfies

1. $\text{Ad} (g_0) = id$ where $id$ is the identity transformation of $T$

2. $\text{Ad} (h_1)\text{Ad} (h_2) = \text{Ad} (h_1h_2)$ for all $h_1, h_2 \in G$

   because $h_1h_2gh_2^{-1}h_1^{-1} = (h_1h_2)g(h_1h_2)^{-1}$

3. Choosing $h_1 = h$, $h_2 = h^{-1}$, we get $\text{Ad} (h^{-1}) = \text{Ad} (h)^{-1}$

This means that the map $h \mapsto \text{Ad} (h)$ is a *linear representation* of the group $G$, i.e. a homomorphism to a group of linear transformations

$$\text{Ad} : G \mapsto GL(n, \mathbb{R})$$

where $n = \text{dim} (G)$.

This representation of $G$ is called *adjoint*.

For commutative Lie groups $G$, e.g. $U(1)$, the adjoint representation $\text{Ad}$ is trivial, i.e. $\text{Ad} (h) = 1 \forall h \in G.$
In terms of local coordinates in a neighbourhood of \( U_0 \) we get the following. Let us denote the inner automorphism of \( G \) determined by \( h \) by \( AD(h) \)

\[
AD(h) : g \mapsto hgh^{-1}, \quad g, h \in G
\]

The corresponding push-forward map is

\[
AD(h)_* : \xi^i \mapsto \eta^i = \frac{\partial \psi^i(\psi(h, x), \varphi(h))}{\partial x^j} \xi^j, \quad \xi \in T_g G, \quad \eta \in T_{hgh^{-1}} G
\]

where \( g \) has local coordinates \((x^1, \ldots, x^n)\).

If \( x = 0 \) then \( g = g_0 \) and both \( \xi, \eta \in T_{g_0} G = T \), and we get \( Ad(h) \)

\[
Ad(h) : \xi^i \mapsto \eta^i = \left. \frac{\partial \psi^i(\psi(h, x), \varphi(h))}{\partial x^j} \right|_{x=0} \xi^j
\]

\[
= \left. \frac{\partial \psi^i(z, \varphi(h))}{\partial z^k} \right|_{z=h} \left. \frac{\partial \psi^j(h, x)}{\partial x^j} \right|_{x=0} \xi^j. \tag{10.14}
\]

This formula can be used to show in particular that

\[
Ad(h_1)Ad(h_2) = Ad(h_1h_2)
\]
Let $F(t) = \exp At$ be a one-parameter subgroup of a Lie group $G$. Then, $\text{Ad} (F(t))$ is a one-parameter subgroup of $GL(n, \mathbb{R})$, and the vector $\frac{d}{dt} \text{Ad} (F(t))|_{t=0}$ lies in the Lie algebra $gl(n, \mathbb{R}) \cong \text{Mat}(n, \mathbb{R})$ of the group $GL(n, \mathbb{R})$ and can be regarded as a linear operator.

This operator is denoted by $\text{ad} A$ and is given by

$$\text{ad} A : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad B \mapsto [A, B], \quad B \in T \cong \mathbb{R}^n$$

The formula is obtained by using (10.14) where we replace $\xi \rightarrow B$, $h \rightarrow f(t)$, differentiate with respect to $t$, set $t = 0$ and use that $\dot{f}^i(0) = A^i$ (prove the formula for $\text{ad} A$).

One-parameter subgroups can be used to define so-called canonical coordinates in a neighbourhood of the identity of a Lie group $G$.

Let $A_1, \ldots, A_n$ form a basis for the Lie algebra $T$.

$$\forall A = \sum A_i x^i \in T \exists \text{ a one-parameter group } F(t) = \exp At.$$ 

To the point $F(1) = \exp A$ we assign as coordinates the coefficients $x^1, \ldots, x^n$ which gives us a system of coordinates in a sufficiently small neighbourhood of $g_0 = 1 \in G$.

These are called the canonical coordinates of the first kind.
Another system of coordinates is obtained by introducing \( F_i(t) = \exp A_i t \) and representing a point \( g \) sufficiently close to \( g_0 \) as

\[
g = F_1(t_1)F_2(t_2) \cdots F_n(t_n)
\]

for small \( t_1, \ldots, t_n \).

Assigning coordinates \( x^1 = t_1, \ldots, x^n = t_n \) to the point \( g \), we get the \textit{canonical coordinates of the second kind}.

**Theorem 3.1.1.** If the functions \( \psi^i(x, y) \) defining the multiplication of points \( x, y \) of a Lie group \( G \) are real analytic (representable by power series) then in some neighbourhood of \( g_0 \in G \) the structure of the Lie algebra of \( G \) determines the multiplication in \( G \).

**Proof.** See the textbook.
**Definition 3.1.3.** A Lie algebra $\mathcal{G} = \{\mathbb{R}^n, c_{jk}^i\}$ is said to be **simple** if it is noncommutative and has no proper ideals, i.e. subspaces $\mathcal{I} \neq \mathcal{G}, 0$ for which $[\mathcal{I}, \mathcal{G}] \subset \mathcal{I}$, and **semisimple** if $\mathcal{G} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_k$ where the $\mathcal{I}_j$ are ideals which are simple as Lie algebras.

These ideals are pairwise commuting $[\mathcal{I}_i, \mathcal{I}_j] = 0$ for $i \neq j$.

A Lie group is defined to be simple or semisimple according to its Lie algebra.

**Definition.** The *Killing form* on an arbitrary Lie algebra $\mathcal{G}$ is defined by

$$\langle A, B \rangle = -\text{tr}(\text{ad } A \text{ ad } B)$$

**Theorem 3.1.4.**

(i) If the Lie algebra $\mathcal{G}$ of a Lie group $G$ is simple, then the linear representation $\text{Ad} : G \mapsto GL(n, \mathbb{R})$ is **irreducible**, i.e. $\mathcal{G}$ has no proper invariant subspaces under the group of inner automorphisms $\text{Ad} (G)$.

(ii) If the Killing form of a Lie algebra is positive definite then the Lie algebra is semisimple.

**Proof.** Blackboard

**Remark.** A stronger result due to Killing and Cartan is

A Lie algebra is semisimple if and only if its Killing form is non-degenerate.
10.3 Linear representations

Definition 3.2.1.

(i) A \textit{linear representation} of a group \( G \) of dim = \( n \) is a homomorphism

\[ \rho : G \mapsto GL(r, \mathbb{R}) \quad \text{or} \quad \rho : G \mapsto GL(r, \mathbb{C}) \]

from \( G \) to a group of real or complex matrices.

(ii) Given a representation \( \rho \) of \( G \), the map

\[ \chi_\rho : G \mapsto \mathbb{R} \quad \text{or} \quad G \mapsto \mathbb{C} \]

declared by

\[ \chi_\rho(g) = \text{tr}\rho(g), \quad g \in G \]

is called the \textit{character} of the representation \( \rho \).

(iii) A representation \( \rho \) of \( G \) is said to be \textit{irreducible} if the vector space \( \mathbb{R}^r \) (or \( \mathbb{C}^r \)) contains no proper subspaces invariant under the matrix group \( \rho(G) \).
Theorem 3.2.2 (Schur’s Lemma).

Let
\[ \rho_i : G \mapsto GL(r_i, \mathbb{R}), \quad i = 1, 2 \]
be two irreps of a group $G$. If $A : \mathbb{R}^{r_1} \mapsto \mathbb{R}^{r_2}$ is a linear transformation changing $\rho_1$ into $\rho_2$, i.e. satisfying
\[ A\rho_1(g) = \rho_2(g)A, \quad \forall g \in G \]
then either $A$ is the zero transformation or else a bijection (in which case $r_1 = r_2$).

**Proof.** Blackboard
If $G$ is a Lie group and a representation $\rho : G \mapsto GL(r, \mathbb{R})$ is a smooth map, then the push-forward map $\rho_*$ is a linear map from the Lie algebra $\mathcal{G} = T_{(1)}$ to the space of all $r \times r$ matrices

$$\rho_* : \mathcal{G} \mapsto Mat(r, \mathbb{R})$$

Verify that $\rho_*$ is a representation of the Lie algebra $\mathcal{G}$, i.e. that it is a Lie algebra homomorphism:

1. It is linear
2. It preserves commutators

$$\rho_* [\xi, \eta] = [\rho_* \xi, \rho_* \eta]$$
**Definition.** A representation

\[ \rho : G \mapsto GL(r, \mathbb{R}) \quad \text{or} \quad \rho : G \mapsto GL(r, \mathbb{C}) \]

is called *faithful* if it is one-to-one, i.e. if its kernel is trivial

\[ \rho(g) \neq I \quad \text{unless} \quad g = g_0 \]

If a Lie group has a faithful representation then it can be realised as a matrix Lie group.

Any matrix Lie group obviously has a faithful representation.

However, not every Lie group can be realised as a matrix Lie group.

One such an example is the group \( \widetilde{SL}(2, \mathbb{R}) \) of all transformations of the real line of the form

\[ x \to x + 2\pi a + \frac{1}{i} \ln \frac{1 - ze^{-ix}}{1 - \bar{z} e^{ix}} , \]

where \( x \in \mathbb{R}, \ a \in \mathbb{R}, \ z \in \mathbb{C}, \ |z| < 1 \) and \( \ln \) is the main branch of the natural logarithmic function, i.e. the continuous branch determined by \( \ln 1 = 0 \).

\( \widetilde{SL}(2, \mathbb{R}) \) is a universal covering group of \( SL(2, \mathbb{R}) \), i.e. it has the same Lie algebra and it is simply connected.
11 Homogeneous Spaces

11.1 Action of a group on a manifold

Definition 5.1.1. We say that a Lie group $G$ is represented as a group of transformations of a manifold $M$, or has a left action on $M$ if

1. there is associated with each of its elements $g$ a diffeomorphism from $M$ to itself

   \[ x \mapsto \mathcal{T}_g(x), \quad x \in M, \]

   such that $\mathcal{T}_{gh} = \mathcal{T}_g \mathcal{T}_h$ for all $g, h \in G$

2. $\mathcal{T}_g(x)$ depends smoothly on the arguments $g, x$, i.e. the map

   \((g, x) \mapsto \mathcal{T}_g(x)\) is a smooth map from $G \times M$ to $M$.

The Lie group $G$ is said to have a right action on $M$ if the above definition is valid with $\mathcal{T}_{gh} = \mathcal{T}_{gh}$ replaced by $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{hg}$.

Example 1. Let $M = G$. Is the action below left or right?

1. $h \mapsto \mathcal{T}_g(h) = gh$, \quad $h \in G$
2. $h \mapsto \mathcal{T}_g(h) = hg$, \quad $h \in G$
3. $h \mapsto \mathcal{T}_g(h) = g^{-1}h$, \quad $h \in G$
4. $h \mapsto \mathcal{T}_g(h) = hg^{-1}$, \quad $h \in G$

Example 2. Any group of real $n \times n$ matrices acts on $\mathbb{R}^n$, e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$
**Definition.** The action of a group $G$ on $M$ is said to be *transitive* if for every two points $x, y$ of $M$ there exists an element $g$ of $G$ such that $\mathcal{T}_g(x) = y$.

**Definition 5.1.2.** A manifold on which a Lie group acts transitively is called a *homogeneous* space of the Lie group.

In particular, $G$ is a homogeneous space for itself, e.g. as $h \mapsto \mathcal{T}_g(h) = gh, \ h \in G$. $G$ is called the *principal* homogeneous space.

**Definition.** Let $x$ be any point of a homogeneous space $M$ of a Lie group $G$. The *isotropy* group (or *stationary* group) $H_x$ of the point $x$ is the stabiliser of $x$ under the action of $G$:

$$H_x = \{h | \mathcal{T}_h(x) = x\}.$$  

**Lemma 5.1.3.** All isotropy groups $H_x$ of points $x$ of a homogeneous space are isomorphic.

**Proof.** Let $x, y$ be any two points of the homogeneous space and $g$ be an element of the Lie group such that $\mathcal{T}_g(x) = y$. The map $H_x \mapsto H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism.

*Prove that $ghg^{-1}$ is an element of $H_y$.*
Theorem 5.1.4. There is a one-to-one correspondence between the points of a homogeneous space $M$ of a group $G$, and the left cosets $gH$ of $H$ in $G$, where $H$ is the isotropy group and $G$ acts on the left.

Proof. Recall that if $G$ is a group, and $H$ is a subgroup of $G$, and $g \in G$, then $gH = \{gh : h \in H\}$ is the left coset of $H$ in $G$ with respect to $g$, while $Hg = \{hg : h \in H\}$ is the right coset.

Let $x_0$ be any point of $M$. Then, we put in correspondence to each left coset $gH_{x_0}$ the point $T_g(x_0) \in M$. This correspondence is independent of the choice of representative of the coset, one-to-one, and onto.

Prove it

Thus, we can write

$$M \cong G/H$$
11.2 Examples of Homogeneous Spaces

1. Sphere:

\[ S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n) \]

2. Real projective space:

\[ \mathbb{R}P^n \cong O(n+1)/(O(1) \times O(n)) \]

3. Torus:

\[ T^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/Z^n \]

4. Stiefel manifolds:

\[ V_{n,k} \cong O(n)/O(n-k) \cong SO(n)/SO(n-k) \]

5. Real Grassmanian manifolds:

\[ G_{n,k} \cong O(n)/(O(k) \times O(n-k)) \]

6. Homogeneous spaces for \( U(n) \)
   
   (a) Sphere:

\[ S^{2n+1} \cong U(n+1)/U(n) \cong SU(n+1)/SU(n) \]

   (b) Complex projective space:

\[ \mathbb{C}P^n \cong U(n+1)/(U(1) \times U(n)) \]

   (c) Complex Grassmanian manifolds:

\[ G_{n,k}^\mathbb{C} \cong U(n)/(U(k) \times U(n-k)) \]
12 Vector Bundles on a Manifold

12.1 Tangent bundle $T(M)$

**Definition.** The *tangent bundle* $T(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T(M)$ are the pairs $(x, \xi)$, $x \in M$ and $\xi \in T_xM$
2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U^T_q$ of $T(M)$ is the set of all pairs $(x, \xi)$ where

$$x = (x^1_q, \ldots, x^n_q) \in U_q \quad \text{and} \quad \xi = \xi^i_q \frac{\partial}{\partial x^i_q} \in T_xM$$

with the local coordinates

$$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, \xi^1_q, \ldots, \xi^n_q) = (x^i_q, \xi^i_q)$$

**Proposition 7.1.1.** The tangent bundle $T(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U^T_q \cap U^T_p$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, \xi^i_p) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^i_p}{\partial x^k_q} \xi^k_p)$$

The Jacobian matrix is

$$
\begin{pmatrix}
\frac{\partial y^\alpha_p}{\partial y^\beta_q} \\
\frac{\partial y^\beta_p}{\partial y^\alpha_q}
\end{pmatrix} =
\begin{pmatrix}
A & 0 \\
H & A
\end{pmatrix}, \quad A = \begin{pmatrix}
\frac{\partial x^i_p}{\partial x^j_q}
\end{pmatrix}, \quad H = \begin{pmatrix}
\frac{\partial^2 x^i_p}{\partial x^j_q \partial x^k_q} \xi^k_p
\end{pmatrix}
$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = (\det A)^2 > 0$$
12.2 Cotangent bundle $T^*(M)$

**Definition.** The *cotangent bundle* $T^*(M)$ of an $n$-dim manifold $M$ is a $2n$-dim manifold defined as follows

1. The points of $T^*(M)$ are the pairs $(x, p)$, $x \in M$, and $p$ is a co-vector at the point $x$: $p \in T^*_x M$

2. Given a chart $U_q$ of $M$ with the local coordinates $(x^i_q)$, the corresponding chart $U^T_q$ of $T^*(M)$ is the set of all pairs $(x, p)$ where

$$x = (x^1_q, \ldots, x^n_q) \in U_q \quad \text{and} \quad p = p_q dx^i_q \in T^*_x M$$

with the local coordinates

$$(y^1_q, \ldots, y^{2n}_q) = (x^1_q, \ldots, x^n_q, p_{q1}, \ldots, p_{qn}) = (x^i_q, p_q^i)$$

**Proposition.** The cotangent bundle $T^*(M)$ is a smooth oriented $2n$-dim manifold.

**Proof.** The transition functions on $U^T_q \cap U^T_p$ are

$$(y^1_p, \ldots, y^{2n}_p) = (x^i_p, p_{pi}) = (x^i_p(x^1_q, \ldots, x^n_q), \frac{\partial x^k_q}{\partial x^i_p} p_{qk})$$

The Jacobian matrix is

$$
\begin{pmatrix}
\frac{\partial y^\alpha_p}{\partial y^\beta_q} \\
\frac{\partial y^{2n}_q}{\partial y^1_p}
\end{pmatrix} = \begin{pmatrix} A & 0 \\ \hat{H} & A^{-1} \end{pmatrix}, \quad A = \left( \frac{\partial x^i_p}{\partial x^j_q} \right), \quad \hat{H} = \left( \frac{\partial^2 x^k_q}{\partial x^i_q \partial x^j_p} p_{qk} \right)
$$

They are smooth, and the Jacobian is

$$J = \det \left( \frac{\partial y^\alpha_p}{\partial y^\beta_q} \right) = 1 > 0$$
The existence of a metric $g_{ij}$ on $M$ gives rise to a map

$$T(M) \hookrightarrow T^*(M) : (x^i, \xi^i) \mapsto (x^i, g_{ij}\xi^j)$$

Since $\omega = p_i dx^i$, a differential one-form on $M$, is invariant under a change of coordinates of $T^*(M)$, it is a differential form on $T^*(M)$.

Its differential

$$\Omega = d\omega = dp_i \wedge dx^i$$

is a nondegenerate closed, $d\Omega = 0$, 2-form on $T^*(M)$.

Thus, $T^*(M)$ is a \textit{symplectic} manifold, i.e. it is equipped with a closed nondegenerate 2-form.
12.3 Normal vector bundle on a submanifold

Let $M$ be an $n$-dim Riemann manifold with metric $g_{ij}$, and let $N$ be a smooth $k$-dim submanifold of $M$. We assume that $N$ is defined by a non-singular system of $(n - k)$ equations.

Recall that the scalar product of two vectors $\xi, \eta \in T_xM$ is given by the metric

$$\langle \xi, \eta \rangle = g_{ij} \xi^i \eta^j$$

Let $x \in N$, and let $\nu \in T_xM$, and let $\nu$ be orthogonal to $N$ at $x$, i.e. orthogonal to the tangent space to $N$ at $x$, which is a $k$-dim subspace of $T_xM$. So, $\nu$ form a $(n - k)$-dim subspace of $T_xM$.

**Definition and Theorem.** The normal vector bundle $\nu_M(N)$ on the submanifold $N$ in $M$ is an $n$-dim submanifold of $T(M)$ defined as

1. The points of $\nu_M(N)$ are the pairs $(x, \nu), x \in N, \nu \in T_xM, \nu \perp N$

2. Given a chart $U$ of $M$ with suitable local coordinates $y^i, i = 1, \ldots, n$, $N$ is defined in $U$ by the equations $y^{k+1} = 0, \ldots y^n = 0$, and $y^1, \ldots, y^k$ serve as local coordinates on $N$.

3. The normal bundle $\nu_M(N)$ is determined as an $n$-dim submanifold of $T(M)$ by the equations

$$y^{k+1} = 0, \ldots y^n = 0, \quad g_{ij} \nu^j = 0, \quad i = 1, \ldots, k$$
Examples.

1. Let $M = \mathbb{R}^n$, and let $N$ be defined by the nonsingular system

$$f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y = (y^1, \ldots, y^n)$$

where $y^i$ are Euclidean coordinates on $\mathbb{R}^n$.

Then, the vectors $\vec{\nabla} f_1, \ldots, \vec{\nabla} f_{n-k}$, are at each point of $N$ perpendicular to $N$ and linearly independent.

Hence,

$$\nu_{\mathbb{R}^n}(N) \cong N \times \mathbb{R}^{n-k}$$
2. More generally, if $N$ is defined as a submanifold of $M$ by a non-singular system

$$f_1(y) = 0, \ldots, f_{n-k}(y) = 0, \quad y \in M$$

then at each point $x \in N$ the vector fields

$$e^i_a(x) = g^{ik}(y) \frac{\partial f_a}{\partial y^k}|_{y=y(x)}, \quad g^{ik}(y)g_{kj}(y) = \delta^i_j, \quad a = 1, \ldots, n-k$$

are linear independent, and for each $\xi \in T_xN$ they satisfy

$$g^{ij}e^i_a(x)\xi^j(x) = \frac{\partial f_a}{\partial y^j}\xi^j(x)|_{y=y(x)} = 0$$

Thus, they are orthogonal to $N$ and any vector normal to $N$ at $x \in N$ has the form

$$\nu = \nu^a e_a(x)$$

The correspondence

$$(x, \nu) \mapsto (x, \nu^1, \ldots, \nu^{n-k})$$

is then a diffeomorphism

$$\nu_M(N) \cong N \times \mathbb{R}^{n-k}$$
3. In particular, if $N$ is the boundary of $\partial A$ of a manifold $A$ with boundary defined by an inequality $f(y) \leq 0$ then $\partial A$ is defined by the single equation $f(y) = 0$, and the normal bundle to the boundary decomposes as a direct product

$$\nu_M(\partial A) \cong \partial A \times \mathbb{R}$$

The bundles we considered are particular cases of smooth fibre bundles, see the textbook.