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Lie Derivative, Covariant Differentiation and the Metric, and the Curvature Tensor

1 Vector Fields and Lie Derivatives

1.1 Vector and Tensor Fields

**Definition.** A vector field is a map that specifies a unique vector at each point \( x \) of the manifold \( M \)

\[
\xi : M \mapsto T(M), \quad x \mapsto \xi_x \in T_x M
\]

A vector field intersects each tangent space of \( T(M) \) at one and only one point, i.e. a vector field is a curve which is nowhere parallel to a tangent space. It is a *cross-section* of \( T(M) \) (*draw a picture*).

In a coordinate basis \((x^i_p)\) we can write

\[
\xi = \xi^i_p(x) \frac{\partial}{\partial x^i}, \quad x \in U_p
\]

Since \( \exists \) a unique vector at each point we drop subscript: \( \xi = \xi^i(x) \frac{\partial}{\partial x^i} \).

A vector field understood as a differential operator maps a scalar function to a scalar function on \( M \)

\[
\xi(f) = \xi^i \frac{\partial f}{\partial x^i}
\]

\( \xi(f) \) can be understood as a map from \( M \) to \( \mathbb{R} \)

\[
\xi(f) : M \mapsto \mathbb{R}, \quad x \mapsto \xi^i(x) \frac{\partial f(x)}{\partial x^i}
\]

Vector fields are linear maps, and satisfy the Leibnitz rule

\[
\xi(fg) = \xi(f) g + f \xi(g)
\]

Thus, vector fields are *derivations* over the smooth real functions.
Tensor Fields

Definition. A tensor field of type \((r, s)\) assigns a unique tensor of type \((r, s)\) to each point \(x\) of the manifold \(M\)

\[(r,s)\xi : M \mapsto T^{(r,s)}(M), \quad x \mapsto (r,s)\xi_x \in T^{(r,s)}_x M\]

It is a cross-section of \(T^{(r,s)}(M)\). In particular, a co-vector field or a field of one-forms is a cross-section of the cotangent bundle \(T^*(M)\).
1.2 The Commutator or Lie bracket

Consider the composition

$$\xi(\eta(f)) = \xi^i \frac{\partial}{\partial x^i} \left( \eta^j \frac{\partial f}{\partial x^j} \right) = \xi^i \eta^j \frac{\partial^2 f}{\partial x^i \partial x^j} + \xi^i \frac{\partial \eta^j}{\partial x^i} \frac{\partial f}{\partial x^j}$$

Because of the first term the composition is not a vector field.

However, the \textit{commutator} or \textit{Lie bracket} defined by

$$[\xi, \eta](f) \equiv \xi(\eta(f)) - \eta(\xi(f)) = \left( \xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial f}{\partial x^j}$$

is a vector field with components

$$[\xi, \eta]^j = \xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}$$

The commutator satisfies

1. \([\xi, \eta] = -[\eta, \xi]\\
2. \([\xi, \eta + \zeta] = [\xi, \eta] + [\xi, \zeta]\\
3. \([\xi, f\eta] = f[\xi, \eta] + \xi(f)\eta\\
4. \([\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0\\

Thus, the vector space of vector fields equipped with the commutator operation is an infinite dimensional Lie algebra.
1.3 Integral Curves

Let $\xi^i(x)$ be a vector field on $M$. Consider the following autonomous system of differential equations

$$\dot{x}^i(t) \equiv \frac{dx^i}{dt} = \xi^i(x^1(t), \ldots, x^n(t)), \quad i = 1, \ldots, n$$

It is called autonomous because the equations have no explicit dependence of $t$.

**Definition.** The solutions $x^i = x^i(t)$ to this system are called the integral curves of the vector field $\xi^i$; the latter is comprised of tangent vectors to the integral curves.

We denote by

$$F_t^i(x^1_0, \ldots, x^n_0) = x^i = x^i(t, x^1_0, \ldots, x^n_0) \quad (1.1)$$

the integral curve of the vector field $\xi^i$ satisfying the initial condition

$$x^i|_{t=t_0} = x^i_0$$

In what follows we always choose $t_0 = 0$, and use the notation

$$x_0 \equiv x^1_0, \ldots, x^n_0$$

Formula (1.1) defines a self-map

$$F_t : (x^1_0, \ldots, x^n_0) \mapsto (x^1(t, x_0), \ldots, x^n(t, x_0))$$

of our manifold, depending on the parameter $t$. 
In mechanics $F_t$ applied to a point $x_0 \in M$ gives the new position of a particle after a time interval $t$, as the particle moves along the integral curve through $x_0$.

It is known from the theory of diff eqs that given any point $x_0$ at which $(\xi^i) \neq 0$ the map $F_t$ is \textit{locally} a diffeomorphism, i.e. $\exists$ a neighbourhood of $x_0$ on which, for sufficiently small $t$, $F_t$ is a diffeomorphism.

Then, for sufficiently small values of the parameters $t$ and $s$ the diffeomorphisms satisfy

$$F_{t+s} = F_t \circ F_s = F_s \circ F_t, \quad F_{-t} = (F_t)^{-1}$$

We say that the diffeomorphisms $F_t$ define a \textit{local group}.

**Definition.** This local abelian one-parameter group of diffeomorphisms $F_t$ is called the \textit{flow} generated by the vector field $\xi^i$.

For small $t$ we have

$$x^i(t, x_0) = x^i_0 + t \xi^i(x_0) + \frac{1}{2} t^2 \frac{\partial \xi^i}{\partial x^j}(x_0) \xi^j(x_0) + O(t^3)$$

Hence, the Jacobian matrix is

$$\frac{\partial x^i}{\partial x^j}_0 = \delta^i_j + t \frac{\partial \xi^i}{\partial x^j} + O(t^2)$$

and inverse

$$\frac{\partial x^i_0}{\partial x^j} = \delta^i_j - t \frac{\partial \xi^i}{\partial x^j} + O(t^2)$$
The above construction can be reversed: Given a one-parameter local group of diffeomorphisms \( F_t = (F^1_t, \ldots, F^n_t) \) we define its velocity field to be the vector field

\[
\xi^i = \left( \frac{d}{dt} F^i_t \right)_{t=0}, \quad i = 1, \ldots, n
\]

**Example.** One-parameter group of rotations through the angle \( t \) about the origin of \( \mathbb{R}^2 \).
1.4 Geometrical Interpretation of the commutator

Let $\xi$ and $\eta$ be vector fields on $M$ with which we associate the flows $F_t$ and $G_s$, respectively.

In general the two flows do not commute, i.e.

$$G_s \circ F_t(x) \neq F_t \circ G_s(x).$$

Here the point $G_s \circ F_t(x)$ is obtained as follows:
Starting with $x \in M$ we move to $F_t(x)$ along the integral curve of $\xi$ followed by moving to $G_s(F_t(x))$ along the integral curve of $\eta$.

On the other hand, to get the point $F_t \circ G_s(x)$, we first move to $G_s(x)$, and then to $F_t(G_s(x))$.

Computing $G_s \circ F_t(x)$ and $F_t \circ G_s(x)$, we get

$$G_s(F_t(x)) - F_t(G_s(x)) = ts[\xi, \eta] + \mathcal{O}(t^3, s^3)$$

Hence, the commutator $[\xi, \eta]$ measures the discrepancy between the points obtained by following the integral curves of $\xi$ and $\eta$ in different orders.
If the Lie bracket of two vector fields vanishes, we say that the vectors commute.

The vectors comprising a coordinate induced basis commute because partial derivatives do.

It turns out that the converse is also true.

If all the elements of a basis for vector fields commute then the basis is coordinate induced.
1.5 The exponential function of a vector field

**Definition.** A one-parameter group of diffeomorphisms $F_t(x)$ with associated vector field $\xi(x)$ is defined to *act on smooth functions* $f = f(x)$ as follows

$$(F_t f)(x) = f(F_t(x)).$$

In particular, the one-parameter group of translations

$$F_t(x) = x + t, \quad \xi = 1$$

of the real line acts as

$$F_t f(x) = f(x + t)$$

For an analytic function $f(x)$ we have

$$F_t f(x) = f(x + t) = f(x) + tf'(x) + \frac{t^2}{2}f''(x) + \cdots$$

$$= \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{d^k}{dx^k} \right) f(x) = \exp \left( t \frac{d}{dx} \right) f(x) = e^{t\partial_x} f(x)$$

**Definition 23.2.1.** The *exponential function* of a vector field $\xi$ is the operator

$$\exp (t \partial_\xi) = 1 + t\partial_\xi + \frac{t^2}{2}(\partial_\xi)^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\partial_\xi)^k$$

where $\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$ is the directional derivation operator in the direction of $\xi$. The action of $e^{t\partial_\xi}$ on functions $f(x)$ is defined by

$$\exp (t \partial_\xi) f(x) = f(x) + t\partial_\xi f(x) + \frac{t^2}{2}(\partial_\xi)^2 f(x) + \cdots$$

\forall t for which this series converges.
Proposition 23.2.2. For analytic vector fields $\xi(x)$ and analytic functions $f(x)$, the exponential function of $\xi(x)$, i.e. $e^{t\partial_t}$, coincides for sufficiently small $t$ with the action of $F_t$ on $f$

$$e^{t\partial_t} f = f(F_t(x))$$

Proof. Let $x = (x^1, \ldots, x^n) \in M$ be a point where $\xi(x) \neq 0$. We can change the coordinates so that in new coordinates $y^i$ the vector field $\xi^i$ becomes

$$\xi = (1, 0, \ldots, 0)$$

on some chart $U \subset M$.

Thus,

$$\exp (t\partial_t) f(x) = f(F_t(x))$$

takes the form

$$\exp \left( t \frac{\partial}{\partial y^1} \right) \tilde{f}(y) = \tilde{f}(y^1 + t, y^2, \ldots, y^n)$$

where

$$\tilde{f}(y) = f(x(y))$$
1.6 The Lie Derivative

We want to define an action of the flow $F_t$ generated by a vector field $\xi = (\xi^i)$ on tensors $T = (T^{i_1 \ldots i_p}_{j_1 \ldots j_q})$ of type $(p, q)$.

We consider a region on which $F_t$ are one-to-one.

\[ \dot{x}^i = \xi^i(x) \Rightarrow y^i(t) = F_t^i(x), \quad y^i(0) = x^i \]

Two pictures of the flow $F_t$

1. $y^i(t)$ is an integral curve of $M$ through the point $x^i$.
   
   Thus, $y^i$ and $x^i$ are different points of the manifold.

   If $t$ is time then the particle moves from $x^i$ to $y^i$.

   Defining an action of $F_t$ on tensors in this picture would require comparing tensors at different points of a manifold.

   We do not know yet how to do this.

2. The point $p$ with coordinates $x$ does not move.

   The local coordinate system in the neighbourhood of $x^i$ changes according to $y^i = F_t^i(x)$.

   This is the point of view we take to define the action of $F_t$ on tensors.
**Definition.** A one-parameter group of diffeomorphisms \( F_t(x) \) with associated vector field \( \xi(x) \) is defined to act on smooth tensors \( T = (T^{i_1...i_p}_{j_1...j_q}) \) of type \((p, q)\) as follows

\[
(F_t T)^{i_1...i_p}_{j_1...j_q}(x) = T^{k_1...k_p}_{l_1...l_q}(y) \frac{\partial y^l_1}{\partial x^{i_1}} \cdots \frac{\partial y^l_{i_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{k_p}},
\]

where

\[
y^i = F_t^i(x)
\]

It is clear from the definition that \((F_t T)^{i_1...i_p}_{j_1...j_q}\) and \(T^{k_1...k_p}_{l_1...l_q}\) are the components of one and the same tensor measured in different coordinate systems.

**Definition 23.3.1.** The Lie derivative of a tensor \( T = (T^{i_1...i_p}_{j_1...j_q}) \) along a vector field \( \xi \) is the tensor \( L_\xi T \) given by

\[
L_\xi T^{i_1...i_p}_{j_1...j_q} = \left[ \frac{d}{dt} (F_t T)^{i_1...i_p}_{j_1...j_q} \right]_{t=0}
\]

Thus, if we regard \( F_t \) as a time-dependent deformation of \( M \) then the Lie derivative measures the rate of change of the tensor \( T = (T^{i_1...i_p}_{j_1...j_q}) \) resulting from this deformation.

Explicit formula for \( L_\xi T \)

\[
L_\xi T^{i_1...i_p}_{j_1...j_q} = \xi^k \frac{\partial T^{i_1...i_p}_{j_1...j_q}}{\partial x^k} + T^{i_1...i_p}_{k_1j_2...j_q} \frac{\partial \xi^k}{\partial x^{i_1}} + T^{i_1...i_p}_{j_1k_2j_3...j_q} \frac{\partial \xi^k}{\partial x^{i_2}} + \cdots + T^{i_1...i_p}_{j_1...j_{q-1}k} \frac{\partial \xi^k}{\partial x^{i_{q-1}}} - T^{i_1...i_p}_{k_1j_1...j_q} \frac{\partial \xi^i_1}{\partial x^k} - T^{i_1...i_p}_{i_1k_2...k_{p-1}j_q} \frac{\partial \xi^i_2}{\partial x^k} - \cdots - T^{i_1...i_{p-1}k}_{j_1...j_q} \frac{\partial \xi^i_{p-1}}{\partial x^k}
\]
**Example 1.** $T$ is a scalar $f$

$$L_\xi f = \xi^k \frac{\partial f}{\partial x^k} = \partial_\xi f$$

The Lie derivative is the directional derivative.

If $L_\xi f = 0$ then $f$ is constant on the integral curves of $\xi$.

Such a function is called an *integral of the field*.

E.g. for $\xi = (-y, x)$ the functions $f(x, y) = x^2 + y^2 - c$ are integral.

**Example 2.** $T$ is a vector field $\eta$

$$L_\xi \eta^i = \xi^k \frac{\partial \eta^i}{\partial x^k} - \eta^k \frac{\partial \xi^i}{\partial x^k} = [\xi, \eta]^i = -L_\eta \xi^i$$

So

$$L_\xi \eta = -L_\eta \xi$$

$$L_\xi (f \eta) = f L_\xi \eta + \eta \partial_\xi f$$

$$L_{[\xi, \eta]} f = \partial_{[\xi, \eta]} f = [\partial_\xi, \partial_\eta] f = [L_\xi, L_\eta] f$$

**HW.** Show that for any tensor $T$, and any vector fields $\xi$ and $\eta$

$$L_{[\xi, \eta]} T = [L_\xi, L_\eta] T$$

Hint: use a coordinate system where $\xi = (1, 0, \ldots, 0)$.
Theorem 23.3.5. Let $\xi_1, \ldots, \xi_m$ be $m$ vector fields in $\mathbb{R}^n$. If $\exists$ a system of coordinates $y^1, \ldots, y^n$ such that at every point the vector $\xi_j$ is tangent to the coordinate axis determined by $y^j$, $j = 1, \ldots, m$ then the vector fields satisfy the condition
\[
[\xi_j, \xi_k] = f^{(1)}_{jk} \xi_j + f^{(2)}_{jk} \xi_k
\]
where $f^{(a)}_{jk}$ are scalars.

Proof. Let $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ be the standard basis vectors in the $(y^i)$ coordinate system. Obviously
\[
[e_j, e_k] = 0.
\]
Tangency of $\xi_j$ to $e_j$ means
\[
\xi_j = f_j(y)e_j, \quad j = 1, \ldots, m
\]
where $f_j$ are scalars.

Taking into account that $[e_j, e_k] = 0$ and $\partial e_i = \frac{\partial}{\partial y^i}$, we get
\[
[\xi_j, \xi_k] = [f_j(y)e_j, f_k(y)e_k] = f_j \frac{\partial f_k}{\partial y^i} e_k - f_k \frac{\partial f_j}{\partial y^k} e_j = f^{(1)}_{jk} \xi_j + f^{(2)}_{jk} \xi_k
\]
where
\[
f^{(1)}_{jk} = -\frac{f_k \partial f_j}{f_j \partial y^k}, \quad f^{(2)}_{jk} = \frac{f_j \partial f_k}{f_k \partial y^j}.
\]
Example 3. Let $T = (T_i)$ be a co-vector (of type $(0, 1)$)

$$L_\xi T_i = \xi^k \frac{\partial T_i}{\partial x^k} + T_k \frac{\partial \xi^k}{\partial x^i}$$

In particular, if $T_i = \frac{\partial f}{\partial x^i} \equiv df_i$ then

$$L_\xi df_i = \xi^k \frac{\partial^2 f}{\partial x^k \partial x^i} + \frac{\partial f}{\partial x^k} \frac{\partial \xi^k}{\partial x^i} = \frac{\partial}{\partial x^i} \left( \xi^k \frac{\partial f}{\partial x^k} \right) = \frac{\partial}{\partial x^i} (L_\xi f)$$

Thus, $L_\xi$ and $d$ commute.

$$L_\xi df = d (L_\xi f)$$
Example 4. Let $T$ be a tensor $(g_{ij})$ of type $(0,2)$

$$L_\xi g_{ij} = \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} \equiv u_{ij}$$

The tensor $u_{ij}$ is called the strain tensor.

If $g_{ij}$ is a metric tensor of $M$ then $u_{ij}$ describes how $g_{ij}$ changes under small deformations $F_t$ defined by the vector field $\xi$.

In particular if the space is Euclidean, $g_{ij} = \delta_{ij}$, then

$$u_{ij} = \frac{\partial \xi^j}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j}, \quad g_{ij} = \delta_{ij}$$
Definition. If \( L_\xi g_{ij} = 0 \) then \( \xi \) is called a Killing vector.

Thus, the metric tensor of \( M \) does not change under small deformations \( F_t \) defined by its Killing vector field \( \xi \).

In a coordinate system \((y^j)\) where a Killing vector field \( \xi \) is \( \xi = (1, 0, \ldots, 0) \), the metric \( g_{ij} \) is independent of \( y^1 \).

**Lemma.** The Killing vector fields of a (pseudo-)Riemann manifold form a Lie algebra with respect to the Lie bracket given by the commutator of two vector fields.

**Proof.** If \( \xi \) and \( \eta \) are two Killing vectors then

\[
L_{[\xi,\eta]} g_{ij} = [L_\xi, L_\eta] g_{ij} = 0
\]

and therefore \([\xi, \eta]\) is a Killing vector too.
Example 5. Let $T$ be the volume element $\sqrt{|g|} \epsilon_{i_1 \ldots i_n}$

$$L_\xi \sqrt{|g|} \epsilon_{i_1 \ldots i_n} = \xi^k \partial \sqrt{|g|} \epsilon_{i_1 \ldots i_n}$$

$$+ \sqrt{|g|} \left( \epsilon_{ki_2 \ldots i_n} \frac{\partial \xi^k}{\partial x^{i_1}} + \cdots + \epsilon_{i_1 \ldots i_{n-1} k} \frac{\partial \xi^k}{\partial x^{i_n}} \right)$$

$$= \sqrt{|g|} \epsilon_{i_1 \ldots i_n} \left( \xi^k \frac{\partial \ln \sqrt{|g|}}{\partial x^k} + \frac{\partial \xi^k}{\partial x^k} \right)$$

$$= \sqrt{|g|} \epsilon_{i_1 \ldots i_n} \left( \xi^k \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial \xi^k}{\partial x^k} \right)$$

On the other hand we have

$$g^{ij} u_{ij} = g^{ij} \left( \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} \right) = \xi^k g^{ij} \frac{\partial g_{ij}}{\partial x^k} + 2 \frac{\partial \xi^k}{\partial x^k}$$

Thus

$$L_\xi \sqrt{|g|} \epsilon_{i_1 \ldots i_n} = \frac{1}{2} g^{ij} u_{ij} \sqrt{|g|} \epsilon_{i_1 \ldots i_n}$$

and in the Euclidean space

$$L_\xi \sqrt{|g|} \epsilon_{i_1 \ldots i_n} = \frac{\partial \xi^k}{\partial x^k} \epsilon_{i_1 \ldots i_n}$$
2 Covariant Differentiation

The differential $d$ transforms a skew-symmetric tensor $T = (T_{i_1...i_k})$ to another skew-symmetric tensor $dT$

$$(dT)_{i_1...i_{k+1}} = \sum_{q=1}^{k+1} (-1)^{q-1} \frac{\partial T_{i_1...\hat{i}_q...i_{k+1}}}{\partial x^{i_q}} = \sum_{q=1}^{k+1} (-1)^{q-1} \partial_{i_q} T_{i_1...\hat{i}_q...i_{k+1}},$$

where $\hat{i}_q$ means that the index $i_q$ is omitted, and

$$\partial_k T_{j_1...j_q}^{i_1...i_p} \equiv \frac{\partial T_{j_1...j_q}^{i_1...i_p}}{\partial x^k}.$$ 

In particular for $k = 1$

$$(dT)_{ij} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j} = \partial_i T_j - \partial_j T_i.$$ 

The difference is important because $\partial_j T_i$ is **not** a tensor.
\( \partial_k T^i \) and \( \partial_k T_i \) transform under arbitrary coordinate changes

\[
x^i = x^i(z^1, \ldots, z^n), \quad i = 1, \ldots, n,
\]
as follows

\[
\partial_q \tilde{T}_j = \frac{\partial \tilde{T}_j}{\partial z^q} = \frac{\partial}{\partial z^q} \left( T_i \frac{\partial x^i}{\partial z^j} \right) = \frac{\partial T_i}{\partial z^q} \frac{\partial x^i}{\partial z^j} + T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j}
\]

\[
= \frac{\partial T_i}{\partial x^p} \frac{\partial^2 x^i}{\partial z^q \partial z^j} + T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j}.
\]

Thus,

\[
\partial_q \tilde{T}_j = \partial_p T_i \frac{\partial x^p}{\partial z^q} \frac{\partial x^i}{\partial z^j} + T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j}, \quad (2.2)
\]

and due to \( T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j} \) it is not a tensor but its skew-symmetric part is.

Similarly for \( \partial_k T^i \) we get

\[
\partial_q \tilde{T}^j = \frac{\partial \tilde{T}^j}{\partial z^q} = \frac{\partial}{\partial z^q} \left( T^i \frac{\partial z^j}{\partial x^i} \right) = \frac{\partial T^i}{\partial z^q} \frac{\partial z^j}{\partial x^i} + T^i \frac{\partial}{\partial x^i} \frac{\partial z^j}{\partial x^i}
\]

\[
= \partial_p T^i \frac{\partial x^p}{\partial z^q} \frac{\partial z^j}{\partial x^i} + T^i \frac{\partial^2 z^j}{\partial x^p \partial x^i} \frac{\partial x^p}{\partial z^q}.
\]

In particular the divergence \( \partial_i T^i \) transforms as

\[
\partial_j \tilde{T}^j = \partial_i T^i + T^i \frac{\partial^2 z^j}{\partial x^p \partial x^i} \frac{\partial x^p}{\partial z^j},
\]

and it is not a scalar.
In Euclidean space with Euclidean coordinates \( x^1, \ldots, x^n \) we need the quantities like \( \partial_k T_{j_1 \ldots j_q}^{i_1 \ldots i_p} \) which transform like tensors.

In \( \mathbb{R}^n \) we introduce the tensor denoted by \( \nabla_k T_{j_1 \ldots j_q}^{i_1 \ldots i_p} \) which is equal to \( \frac{\partial T_{j_1 \ldots j_q}^{i_1 \ldots i_p}}{\partial x^k} \) in Euclidean coordinates.

In any system with coordinates \( z^1, \ldots, z^n \) its components are

\[
\nabla_r \tilde{T}^{(k)}_{(l)} = \nabla_s T^{(i)}_{(j)} \frac{\partial x^s}{\partial z^r} \frac{\partial x^{(j)}}{\partial z^{(l)}} \frac{\partial z^{(k)}}{\partial x^{(i)}},
\]

where \( (k) = k_1 \ldots k_p \), \( (l) = l_1 \ldots l_q \), and so on, and

\[
\frac{\partial x^{(j)}}{\partial z^{(l)}} = \frac{\partial x^{j_1}}{\partial z^{l_1}} \cdots \frac{\partial x^{j_q}}{\partial z^{l_q}}, \quad \frac{\partial z^{(k)}}{\partial x^{(i)}} = \frac{\partial z^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial z^{k_p}}{\partial x^{i_p}},
\]

\[
\nabla_s T^{(i)}_{(j)} = \frac{\partial T_{j_1 \ldots j_q}^{i_1 \ldots i_p}}{\partial x^s} = \partial_s T^{(i)}_{(j)}.
\]

Notation

\[
T_{j_1 \ldots j_q; k}^{i_1 \ldots i_p} \equiv \nabla_k T_{j_1 \ldots j_q}^{i_1 \ldots i_p}.
\]
For simplicity consider \((T^i)\) and \((T_i)\).

We have (note that \(\tilde{T}^k = T^i \frac{\partial z^k}{\partial x^i}\) and \(T^i = \tilde{T}^k \frac{\partial x^i}{\partial z^k}\))

\[
\nabla_r \tilde{T}^k = \partial_s T^i \frac{\partial x^s}{\partial z^r} \frac{\partial z^k}{\partial x^i} = \frac{\partial}{\partial z^r} \left( T^i \frac{\partial z^k}{\partial x^i} \right) - T^i \frac{\partial}{\partial z^r} \frac{\partial z^k}{\partial x^i}
\]

\[
= \frac{\partial}{\partial z^r} \left( T^i \frac{\partial z^k}{\partial x^i} \right) - T^i \frac{\partial}{\partial z^r} \frac{\partial z^k}{\partial x^i}
\]

\[
= \frac{\partial \tilde{T}^k}{\partial z^r} - \tilde{T}^s \frac{\partial x^i}{\partial z^s} \frac{\partial^2 z^k}{\partial x^m \partial x^i} \frac{\partial x^m}{\partial z^r}.
\]

Introducing

\[
\Gamma_{sr}^k = - \frac{\partial x^i}{\partial z^s} \frac{\partial x^m}{\partial z^r} \frac{\partial^2 z^k}{\partial x^m \partial x^i}, \tag{2.3}
\]

we get

\[
\nabla_r \tilde{T}^k = \frac{\partial \tilde{T}^k}{\partial z^r} + \Gamma_{sr}^k \tilde{T}^s.
\]

Thus, we have proven

**Theorem 28.1.2.** Let \((T^i)\) be a vector field, and let \(\nabla_k T^i\) be a tensor given in terms of Euclidean coordinates \(x^1, \ldots, x^n\) by the formula \(\nabla_k T^i = \frac{\partial T^i}{\partial x^k}\). Then, in arbitrary coordinates \(z^1, \ldots, z^n\) the transformed components \(\nabla_k \tilde{T}^i\) are given by the formula

\[
\nabla_r \tilde{T}^k = \frac{\partial \tilde{T}^k}{\partial z^r} + \Gamma_{sr}^k \tilde{T}^s,
\]

where the coefficients \(\Gamma_{sr}^k\) are defined in (2.3).
Similarly, we have

**Theorem 28.1.3.** Let \((T_i)\) be a covector field, and let \(\nabla_k T_i\) be a tensor given in terms of Euclidean coordinates \(x^1, \ldots, x^n\) by the formula \(\nabla_k T_i = \frac{\partial T_i}{\partial x^k}\). Then, in arbitrary coordinates \(z^1, \ldots, z^n\) the transformed components \(\nabla_k \tilde{T}_i\) are given by the formula

\[
\nabla_r \tilde{T}_k \equiv \tilde{T}_k ; r = \frac{\partial \tilde{T}_k}{\partial z^r} - \Gamma^s_{kr} \tilde{T}_s.
\]

**Proof.** We have (note that \(\tilde{T}_k = T_i \frac{\partial x^i}{\partial z^k}\) and \(T_i = \tilde{T}_k \frac{\partial z^k}{\partial x^i}\))

\[
\nabla_r \tilde{T}_k = \partial_s T_i \frac{\partial x^s}{\partial z^r} \frac{\partial x^i}{\partial z^k} = \frac{\partial T_i}{\partial z^r} \frac{\partial x^i}{\partial z^k} - \frac{\partial T_i}{\partial z^r} \frac{\partial x^i}{\partial z^k}
\]

\[
= \frac{\partial}{\partial z^r} \left( T_i \frac{\partial x^i}{\partial z^k} \right) - T_i \frac{\partial^2 x^i}{\partial z^r \partial z^k}
\]

\[
= \frac{\partial \tilde{T}_k}{\partial z^r} - \tilde{T}_i \frac{\partial z^s}{\partial x^i} \frac{\partial^2 x^i}{\partial z^r \partial z^k} = \frac{\partial \tilde{T}_k}{\partial z^r} - \Gamma^s_{kr} \tilde{T}_s,
\]

where we have used

\[
\frac{\partial z^s}{\partial x^i} \frac{\partial^2 x^i}{\partial z^r \partial z^k} = \frac{\partial}{\partial z^r} \left( \frac{\partial z^s}{\partial x^i} \frac{\partial x^i}{\partial z^k} \right) - \left( \frac{\partial}{\partial z^r} \frac{\partial z^s}{\partial x^i} \right) \frac{\partial x^i}{\partial z^k}
\]

\[
= \frac{\partial}{\partial z^r} \delta^s_k - \frac{\partial^2 z^s}{\partial x^m \partial x^i} \frac{\partial x^m}{\partial z^r} \frac{\partial x^i}{\partial z^k} = \Gamma^s_{kr}.
\]
Generalising the computations above, we get

**Theorem 28.1.4.** Let $T_{(j)}^{(i)}$ be the components of a tensor of type $(p, q)$, and let $\nabla_k T_{(j)}^{(i)}$ be a tensor given in terms of Euclidean coordinates $x^1, \ldots, x^n$ by the formula

$$\nabla_k T_{(j)}^{(i)} = \frac{\partial T_{(j)}^{(i)}}{\partial x^k}.$$  

Then, in arbitrary coordinates $z^1, \ldots, z^n$ the transformed components $\nabla_r \tilde{T}_{(l)}^{(k)}$ are given by the formula

$$\nabla_r \tilde{T}_{(l)}^{(k)} \equiv \tilde{T}_{(l);r}^{(k)} = \frac{\partial \tilde{T}_{(l)}^{(k)}}{\partial z^r} + \sum_{a=1}^{p} \Gamma_{sr}^{ka} \tilde{T}_{l_1 \ldots l_q}^{k_1 \ldots k_p} - \sum_{a=1}^{q} \Gamma_{la}^{s} \tilde{T}_{l_1 \ldots (l_a \rightarrow s) \ldots l_q}^{k_1 \ldots k_p} \tag{2.4}$$

For zero-rank tensors (functions or scalars) $T$, we get $\nabla_r T = \partial_r T$.

For second-rank tensors we get

$$\nabla_r \tilde{T}_l^{k} \equiv \tilde{T}_{l;r}^{k} = \frac{\partial \tilde{T}_l^{k}}{\partial z^r} + \Gamma_{sr}^{k} \tilde{T}_s^{l} - \Gamma_{lr}^{s} \tilde{T}_l^{s},$$

$$\nabla_r \tilde{T}_{lm} \equiv \tilde{T}_{lm;r} = \frac{\partial \tilde{T}_{lm}}{\partial z^r} - \Gamma_{lr}^{s} \tilde{T}_{sm} - \Gamma_{mr}^{s} \tilde{T}_l^{s},$$

$$\nabla_r \tilde{T}_{km} \equiv \tilde{T}_{km;r} = \frac{\partial \tilde{T}_{km}}{\partial z^r} + \Gamma_{sr}^{k} \tilde{T}_s^{m} + \Gamma_{sr}^{m} \tilde{T}_k^{s}.$$
Let us now determine how $\Gamma_{ij}^k$ transform under an arbitrary coordinate change $z^i = z^i(z')$, $i = 1, \ldots, n$.

In what follows to simplify the notations we use the convention

$$z'^i \equiv z^i', \quad i = 1, \ldots, n, \quad \text{e.g. } z'^2 = z'^2.$$

Then, we have

$$\nabla_k \tilde{T}_i = \frac{\partial \tilde{T}_i}{\partial z^k} - \Gamma_{ik}^r \tilde{T}_r, \quad \text{in } z \text{ coordinates},$$

$$\nabla_{k'} \tilde{T}'_{i'} = \frac{\partial \tilde{T}'_{i'}}{\partial z'^{k'}} - \Gamma_{i'k'}^{r'} \tilde{T}'_r', \quad \text{in } z' \text{ coordinates}.$$

Since they are tensors

$$\nabla_{k'} \tilde{T}'_{i'} = \nabla_k \tilde{T}_i \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k} = \partial_k \tilde{T}_i \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k} - \Gamma_{ik'}^r \tilde{T}_r \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k}.$$

On the other hand eq. (2.2) shows how to express $\partial_{k'} \tilde{T}'_{i'}$ in terms of $\partial_k \tilde{T}_i$

$$\nabla_{k'} \tilde{T}'_{i'} = \frac{\partial \tilde{T}'_{i'}}{\partial z'^{k'}} - \Gamma_{i'k'}^{r'} \tilde{T}'_r = \partial_k \tilde{T}_i \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k} + \tilde{T}_r \frac{\partial^2 z^r}{\partial z'^{k'} \partial z'^{k'}} - \Gamma_{i'k'}^{r'} \tilde{T}'_r \frac{\partial z^r}{\partial z'^{k'}}.$$

Comparing the two expressions, we get

$$\Gamma_{i'k'}^{r'} \frac{\partial z^r}{\partial z'^{k'}} = \Gamma_{ik}^r \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k} + \frac{\partial^2 z^r}{\partial z'^{k'} \partial z'^{k'}}.$$

Multiplying by $\frac{\partial z'^{s'}}{\partial z^r}$, and summing over $r$, we get

$$\Gamma_{i'k'}^{s'} = \Gamma_{ik}^{r'} \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^k} + \frac{\partial^2 z^r}{\partial z'^{k'} \partial z'^{k'}} \frac{\partial z'^{s'}}{\partial z^r}.$$

(2.5)
This formula motivates the general definition

**Def 28.1.5.** An operation of covariant differentiation of tensors is said to be defined if we are given, in terms of any system of coordinates \( z^1, \ldots, z^n \) a family of functions \( \Gamma^k_{pq}(z) \) which transform under arbitrary coordinate changes \( z = z(z') \) according to the formula

\[
\Gamma^s'_{i'k'} = \Gamma^r_{ik} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}} \frac{\partial z^{s'}}{\partial z^r} + \frac{\partial^2 z^r}{\partial z^{i'} \partial z^{k'}} \frac{\partial z^{s'}}{\partial z^r}.
\]

The quantities \( \Gamma^k_{pq}(z) \) are called Christoffel symbols.

The covariant derivatives of tensors are given by

\[
\nabla_r \tilde{T}^{(k)}_{(l);r} = \frac{\partial \tilde{T}^{(k)}_{(l)}}{\partial z^r} + \sum_{a=1}^p \Gamma^s_{lr} \tilde{T}^{k_1\ldots(k_a\rightarrow s)\ldots k_p}_{l_1\ldots l_q} - \sum_{a=1}^q \Gamma^s_{lr} \tilde{T}^{k_1\ldots k_p}_{l_1\ldots(l_a\rightarrow s)\ldots l_q}
\]

which for zero-rank tensors (functions or scalars) \( T \) coincide with partial derivatives \( \nabla_r T = \partial_r T \),

and for vectors and covectors take the form

\[
\nabla_k T^i = \frac{\partial T^i}{\partial z^k} + \Gamma^i_{rk} T^r, \quad \nabla_k T_i = \frac{\partial T_i}{\partial z^k} - \Gamma^r_{ik} T_r. \quad (2.6)
\]

**Def.** An operation of covariant differentiation is called a connection.
Def. A connection is said to be *Euclidean or affine* if there exist coordinates $x^1, \ldots, x^n$ in terms of which $\Gamma^k_{ij} = 0$.

Such coordinates are often called Euclidean or affine.

Christoffel symbols are not components of a tensor. However,

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = \Gamma^i_{[jk]},$$

is a tensor. It is called the *torsion tensor*.

**Def 28.1.5.** A connection $\Gamma^i_{jk}$ is said to be *symmetric or torsion-free* if the torsion tensor is identically zero, i.e. if $\Gamma^i_{jk} = \Gamma^i_{kj}$.

For example, if the connection is affine then in Euclidean coordinates $\Gamma^i_{jk} = 0$, and therefore $T^i_{jk} = 0$. 

Two main properties of the operation of covariant differentiation are

1. It is linear, and commutes with the operation of contraction.

The linearity is obvious, and to show the commutativity it is sufficient to consider tensors of rank $(1, 1)$.

Since $T = T^k_k$ is a scalar $\nabla_r T = \partial_r T$. Then we get

$$\nabla_r T^k_k = \frac{\partial T^k_k}{\partial z^r} + \Gamma^k_{sr} T^s_k - \Gamma^s_{kr} T^k_s = \partial_r T.$$ 

2. The covariant derivative of a product $T^{(i)(j)}_{(p)(q)} = R^{(i)}_{(p)} S^{(j)}_{(q)}$ of tensors is calculated using the usual Leibniz product rule

$$\nabla_k T^{(i)(j)}_{(p)(q)} = (\nabla_k R^{(i)}_{(p)}) S^{(j)}_{(q)} + R^{(i)}_{(p)} (\nabla_k S^{(j)}_{(q)}).$$

It follows from (2.4), and the product rule for usual derivatives.
In fact these properties together with the formulae for the covariant derivatives of scalars, vectors and covectors uniquely determine the tensor operation of covariant differentiation corresponding to a given connection $\Gamma^i_{jk}$.

**Theorem 28.2.6.** Let $\Gamma^i_{jk}$ be a connection. If a tensor operation $\nabla_k$ satisfies the following four conditions

1. It is linear, and commutes with the operation of contraction,
2. On zero-rank tensors (functions) $T$ the operation coincides with the partial derivative $\nabla_k T = \partial_k T$,
3. On vectors and covectors the operation is given by
   
   \[ \nabla_k T^i = \frac{\partial T^i}{\partial z^k} + \Gamma^i_{rk} T^r, \quad \nabla_k T_i = \frac{\partial T_i}{\partial z^k} - \Gamma^r_{ik} T_r, \]
4. On a product $T^{(i)(j)}_{(p)(q)} = R^{(i)}_{(p)} S^{(j)}_{(q)}$ of tensors the operation acts according to the Leibniz rule
   
   \[ \nabla_k T^{(i)(j)}_{(p)(q)} = (\nabla_k R^{(i)}_{(p)}) S^{(j)}_{(q)} + R^{(i)}_{(p)} (\nabla_k S^{(j)}_{(q)}), \]

then it is the operation of covariant differentiation determined by the $\Gamma^i_{jk}$, and on general tensors it is given by (2.4).

**Proof.** Let $e_1, \ldots, e_n$ be the standard basis of vector fields, and let $e^1, \ldots, e^n$ be the dual basis of covector fields.

The components of these tensors are $(e_i)^j = \delta^j_i = (e^j)_i$. 

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For a vector $\xi = \xi^i e_i$ we have

$$\nabla_k \xi = \left( \frac{\partial \xi^a}{\partial x^k} + \Gamma^a_{jk} \xi^j \right) e_a,$$

and therefore if $\xi = e_i$ we get

$$\nabla_k e_i = \Gamma^a_{ik} e_a.$$

Analogously,

$$\nabla_k e^i = -\Gamma^i_{ak} e^a.$$

These equations can be viewed as defining $\Gamma^i_{jk}$.

Every tensor $T$ with components $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ has the form

$$T = T^{i_1 \ldots i_p}_{j_1 \ldots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}.$$

Here $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ should be considered as scalars because $(e_i)$ and $(e^j)$ transform under the action of $\nabla_k$.

Consider for simplicity tensors of type $(0, 2)$.

Then, $T = T_{ij} e^i \otimes e^j$, and by using the conditions (1-4), we get

$$\nabla_k (T) = \nabla_k (T_{ij}) e^i \otimes e^j + T_{ij} (\nabla_k e^i) \otimes e^j + T_{ij} e^i \otimes (\nabla_k e^j)$$

$$= \partial_k T_{ij} e^i \otimes e^j - T_{lj} \Gamma^i_{lk} e^a \otimes e^j - T_{il} \Gamma^j_{lk} e^a$$

$$= (\partial_k T_{ij} - T_{lj} \Gamma^i_{lk} - T_{il} \Gamma^j_{lk}) e^i \otimes e^j.$$

Thus, the components of $\nabla_k T$ have the form

$$\partial_k T_{ij} - T_{lj} \Gamma^i_{ik} - T_{il} \Gamma^j_{jk},$$

which proves the theorem for tensors of type $(0, 2)$. 

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3 Parallel Transport of Tensor Fields

Let \( \xi \in T_PM, \ P \in M, \ \dim M = n, \) and let \( T = (T_{(i)}^{(j)}), \)
\( (i) \equiv i_1 \ldots i_p, \ (j) \equiv j_1 \ldots j_q \) be a tensor of type \((p, q)\).

**Def.** The *directional derivative* of \( T \) at \( P \) along (or relative to) the vector \( \xi \)
\[ \nabla_\xi T_{(j)}^{(i)} = \xi^k \nabla_k T_{(j)}^{(i)} \]
is a tensor of the same type \((p, q)\).

For a scalar \( f \) (rank-zero tensor)
\[ \nabla_\xi f = \xi^k \partial_k f = \partial_\xi f, \]
it coincides with the directional derivative of a function.
Let $\xi(t)$ be a velocity vector of some curve $C$:

$$x^i = x^i(t), \quad \xi^i(t) = \frac{dx^i}{dt}, \quad i = 1, \ldots, n.$$ 

If $\partial_\xi f = 0$ for all points of $C$ then

$$f(x^1(t), \ldots, x^n(t)) = \text{const}.$$ 

The question whether a vector (or a tensor field) is constant along $C$ is meaningless because a tensor of rank $> 0$ has different components in different coordinates.

Thus, even if the tensor components are constant in one system of coordinates they are not in another (unless it is the zero tensor).

A given connection (covariant differentiation) provides us with a way to compare two tensors attached to different points of $M$. 
Def 29.1.1. Let $\Gamma^i_{kj}$ be a connection defined on a manifold $M$ with local coordinates $x^1, \ldots, x^n$ (in some chart $U$), and let $x^i(t), a \leq t \leq b$, be a segment $C$ of an arbitrary curve. We say that a tensor field $T$ is \textit{covariantly constant} or parallel along $C$ if

$$\nabla_\xi T = \xi^k \nabla_k T = 0, \quad a \leq t \leq b, \quad \xi^k = \frac{dx^k}{dt}.$$ 

For vector fields one gets

$$\nabla_\xi T^i = \xi^k \nabla_k T^i = \xi^k \left( \frac{\partial T^i}{\partial x^k} + \Gamma^i_{jk} T^j \right) = 0.$$ 

\textbf{Remarks}

1. The concept of parallelism is \textit{coordinate independent} because covariant differentiation is a tensor operation.

2. The concept of parallelism \textit{depends on the connection} given.

3. In general the concept of parallelism \textit{depends on $C$}.

4. However if the connection is \textit{Euclidean (or affine)} and if

$$\nabla_\xi T_{(i)}^{(j)} = 0 \text{ along } C \text{ then since in affine coordinates } \Gamma^i_{jk} = 0,$$

the condition $\nabla_\xi T_{(i)}^{(j)} = \xi^k \partial_k T_{(i)}^{(j)} = 0$ implies that

$T_{(i)}^{(j)} = \text{const along } C.$

Extending $T_{(i)}^{(j)} = \text{const}$ to all points of chart $U$, we get a tensor field parallel along any other curve in chart $U$. 
5. This provides a relation to the fifth postulate of Euclid:

“Given a line through a point $P$, and a point $Q$ not on the line, there is exactly one line through $Q$ parallel to the given line”.

This can be rephrased as

“In Euclidean geometry for each nonzero vector $(T^i)_P$ at the point $P$ there one (up to scalar multiples) and only one parallel vector at any point $Q$”.

In affine coordinates a parallel vector has the same components (up to scalar multiples) as the vector $(T^i)_P$.

6. A product of two covariantly constant tensors $T = (T^{(i)}_{(j)})$ and $S = (S^{(k)}_{(l)})$ is covariantly constant

$$\nabla_\xi (T^{(i)}_{(j)}S^{(k)}_{(l)}) = (\nabla_\xi T^{(i)}_{(j)})S^{(k)}_{(l)} + T^{(i)}_{(j)}(\nabla_\xi S^{(k)}_{(l)}) = 0.$$
If the connection is not Euclidean then

**What do we mean when we say that two vectors (or tensors), one at each of two distinct points, are parallel?**

To compare the vectors we need to move one vector to the second one. This is done by means of “**parallel transport**”.

**Def 29.1.2.** Let \((T^i)_P\) be a vector at a point \(P(x_1^0, \ldots, x_n^0)\) and let \(x^i(t), 0 \leq t \leq 1\), be a curve segment \(C\) joining \(P\) to a point \(Q(x_1^1, \ldots, x_n^1)\). The unique vector field \((T^i)\) defined at all points of \(C\), taking value \((T^i)_P\) at \(P(t = 0)\), and parallel along \(C\) is said to result from **parallel transport** of the vector \((T^i)_P\) along \(C\) to \(Q(t = 1)\).

The value of the field \((T^i)_P\) at \(Q\) is denoted by \((T^i)_Q\), and is called the **result of parallel transport** of \((T^i)_P\) along \(C\) from \(P\) to \(Q\), relative to the given connection.

The vector field \((T^i)\) is determined by the equation of parallel transport

\[
\frac{dx^k}{dt} \nabla_k T^i = \frac{dx^k}{dt} \left( \frac{\partial T^i}{\partial x^k} + \Gamma^{i}_{jk} T^j \right) = \frac{dT^i}{dt} + \left( \frac{dx^k}{dt} \Gamma^{i}_{jk} \right) T^j = 0 ,
\]

and the initial conditions

\[
T^i(0) = T^i, \quad i = 1, \ldots, n .
\]

As we saw in Euclidean geometry the result of parallel transport of a vector from one point to another is independent of the path.
4 Geodesics

*Given an arbitrary connection, which curves play the role of “straight lines”?*

**Def 29.2.1.** A curve $x^i = x^i(t)$ is called *geodesics* if the vector field defined by its tangent vector $T^i = dx^i/dt$ is parallel along the curve itself, i.e. if the curve parallel transports its own tangent vector:

$$\nabla_{T}(T) = \nabla_{\dot{x}}(\dot{x}) = 0 .$$

In components one gets

$$\nabla_{T}(T)^i = \frac{dx^k}{dt} \nabla_k \left( \frac{dx^i}{dt} \right) = \frac{d}{dt} \frac{dx^i}{dt} + \left( \frac{dx^k}{dt} \Gamma^i_{jk} \right) \frac{dx^j}{dt} = 0 ,$$

and the equations for the geodesics take the form

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 , \quad i = 1, \ldots, n .$$

**Remarks**

1. If $\Gamma^i_{jk} = 0$ then the solutions are just straight lines.

2. The geodesics *do not depend on the torsion*. They depend only on the symmetric part $\Gamma^i_{(jk)} = \Gamma^i_{jk} + \Gamma^i_{kj}$ of the connection.

3. The equations are *second-order ordinary differential equations*. They can be interpreted as equations of motion of a freely moving particle on the manifold $M$ (to be discuss later).
4. The system has a **unique solution** satisfying the initial conditions

\[ x^i|_{t=0} = x_0^i, \quad \frac{dx^i}{dt}|_{t=0} = v_0^i, \quad i = 1, \ldots, n, \]

for each choice of \( x_0^i, v_0^i \).

To be precise

**Theorem 29.2.2.** Let \( \Gamma^i_{kj} \) be a connection defined on a manifold \( M \). Then for any point \( P \) in some chart \( U \) and any vector \( (T^i)_P \) attached to the point there exists a unique geodesic starting from \( P \) and with the initial tangent vector \( (T^i)_P \).
5 Connections Compatible with the Metric

In general a (pseudo-)Riemann metric and a connection defined on a manifold $M$ are completely independent.

It would however be better if they were related because

1. A given connection can be used to parallel transport a metric defined at a point $P$ to any point $Q$ of $M$.
   
   We want the results to be independent of the curves used.

2. A metric can be used to lower indices of a tensor.
   
   We want the covariant differentiation to commute with the operation of lowering indices.

3. The distance is determined by the metric.
   
   We want the set of geodesics connecting two nearby points to contain the shortest curve.
**Def 29.2.1.** A connection $\Gamma^i_{kj}$ is said to be *compatible with a metric* $g_{ij}$ if the covariant derivative of the metric tensor $(g_{ij})$ is identically zero:

$$\nabla_k g_{ij} \equiv 0, \quad i, j, k = 1, \ldots, n.$$ 

**Remarks.**

For a given connection compatible with the metric on a manifold

1. The corresponding operation of covariant differentiation *commutes* with the operation of lowering any index of a tensor

   $$\nabla_k (g_{lm} T_{(j)}^{(i)}) = (\nabla_k g_{lm}) T_{(j)}^{(i)} + g_{lm} (\nabla_k T_{(j)}^{(i)}) = g_{lm} (\nabla_k T_{(j)}^{(i)}).$$

2. If vector fields $T^i(t)$ and $S^i(t)$ are both parallel along a curve $x^i = x^i(t)$, then their *scalar product is constant* along the curve

   $$\frac{d}{dt} \langle T, S \rangle = \frac{d}{dt} (g_{ij} T^i S^j) = \frac{d x^k}{dt} \nabla_k (g_{ij} T^i S^j) = g_{ij} \frac{d x^k}{dt} \nabla_k (T^i S^j) = 0.$$ 

   In other words

   *Parallel transport of vectors from a point $P$ to a point $Q$ along a given curve defines an orthogonal transformation from the tangent space at $P$ to the tangent space at $Q.*
6 Symmetric $\Gamma^i_{kj}$ Compatible with the Metric

**Theorem 29.3.2.** If the metric $g_{ij}$ is non-singular, $\det(g_{ij}) \neq 0$, on a chart $U$ of the manifold $M$ under consideration, then there is a unique torsion-free (symmetric) connection which is compatible with the metric. It is given in any system of coordinates $x^1, \ldots, x^n$ by Christoffel’s formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

**Proof.** We have

$$\nabla_k g_{ij} = \partial_k g_{ij} - g_{pj} \Gamma^p_{ik} - g_{ip} \Gamma^p_{jk} = \partial_k g_{ij} - \Gamma_{j,ik} - \Gamma_{i,jk} = 0,$$

$$\Rightarrow \quad \partial_j g_{ki} - \Gamma_{i,kj} - \Gamma_{k,ij} = 0,$$

$$\Rightarrow \quad \partial_i g_{jk} - \Gamma_{k,ji} - \Gamma_{j,ki} = 0,$$

where $\Gamma_{j,ik} \equiv g_{pj} \Gamma^p_{ik} = g_{jp} \Gamma^p_{ik}$. Taking the sum of the last two equations and subtracting from the sum the first equation, we get

$$\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij} - 2\Gamma_{k,ij} = 0,$$

where we took into account that the connection is torsion-free, and therefore $\Gamma_{k,ij} = \Gamma_{k,ji}$. Thus, one finds

$$\Gamma_{k,ij} = \frac{1}{2} \left( \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right) \Rightarrow \quad \Gamma^k_{ij} = \frac{1}{2} g^{kp} (\partial_k g_{pj} + \partial_j g_{ip} - \partial_p g_{ij}).$$
Remarks. We consider symmetric and compatible connections

1. If $\partial_i g_{jk} = 0$ at a given point for any $i, j, k = 1, \ldots, n$, then at that point all $\Gamma^k_{ij} = 0$.

Example. Consider a surface $\sigma$

$$x^1 = x^1(z^1, z^2), \quad x^2 = x^2(z^1, z^2), \quad x^3 = x^3(z^1, z^2),$$

in $\mathbb{R}^3$ with Euclidean coordinates $x^i$. At a given point $P$ we introduce new Euclidean coordinates $x, y, z$ so that the $z$-axis is perpendicular to the surface at $P$ while the $x$-axis and the $y$-axis are tangent to the surface at $P$. Then in some neighbourhood of $P$ the surface $\sigma$ is given by

$$z = f(x, y), \quad \frac{\partial f}{\partial x}|_{(0,0)} = \frac{\partial f}{\partial y}|_{(0,0)} = 0.$$

The induced metric is given by

$$ds^2_\sigma = ds^2_{\mathbb{R}^3}|_{z=f(x,y)} = dx^2 + dy^2 + df(x, y)^2 = g_{ij}dz^idz^j, \quad z^1 = x, \quad z^2 = y,$$

where

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j}.$$

Thus at $P$ where $\frac{\partial f}{\partial z^i} = 0$ we have $g_{ij} = \delta_{ij}$, and

$$\frac{\partial g_{ij}}{\partial z^k} = \frac{\partial}{\partial z^k} \left( \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j} \right) = \left( \frac{\partial}{\partial z^k} \frac{\partial f}{\partial z^i} \right) \frac{\partial f}{\partial z^j} + \frac{\partial f}{\partial z^i} \left( \frac{\partial}{\partial z^k} \frac{\partial f}{\partial z^j} \right) = 0.$$

Thus at $P$ the symmetric and compatible connection is zero.
2. The divergence of a vector field

\[
\text{div } T = \nabla_i T^i = \frac{\partial T^i}{\partial x^i} + \Gamma^i_{ki} T^k,
\]

can be expressed as \((g = \det(g_{ij}))\)

\[
\nabla_i T^i = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} T^i \right),
\]

by using

\[
\Gamma^i_{ki} = \frac{1}{2} g^{jp} \left( \partial_k g_{pi} + \partial_i g_{kp} - \partial_p g_{ki} \right) = \frac{1}{2} g^{jp} \partial_k g_{pi} = \frac{1}{2} g \partial_k g.
\]

Thus

\[
\sqrt{|g|} \nabla_i T^i = \frac{\partial}{\partial x^i} \left( \sqrt{|g|} T^i \right),
\]

and therefore if \(M\) is an oriented closed manifold then

\[
\int_M \nabla_i T^i \Omega_M = \int_M \nabla_i T^i \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n = 0.
\]
7 Geodesics and the Metric

The length of a curve $C: x^i = x^i(t), a \leq t \leq b$ between the points $P$ and $Q$ is given by

$$L = \int_{C} ds = \int_{a}^{b} \sqrt{g_{ij} \dot{x}^i \dot{x}^j} \, dt = \int_{a}^{b} \mathcal{L}(x, \dot{x}) \, dt. \quad (7.7)$$

The shortest curve satisfies equations which are obtained by considering the variation of $L$ and setting it to 0

$$\delta L = \int_{a}^{b} \delta \mathcal{L} \, dt = \int_{a}^{b} \left( \frac{\partial \mathcal{L}}{\partial x^k} \delta x^k + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \delta \dot{x}^k \right) \, dt$$

$$= \int_{a}^{b} \frac{\partial \mathcal{L}}{\partial x^k} \delta x^k \, dt - \int_{a}^{b} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \right) \delta x^k \, dt,$$

where we integrated by parts, and used $\delta x^k(a) = \delta x^k(b) = 0$. Thus we get the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^k} - \frac{\partial \mathcal{L}}{\partial x^k} = 0.$$

We then have

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{\mathcal{L}} g_{lk} \ddot{x}^l, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{1}{2\mathcal{L}} \partial_k g_{lj} \dot{x}^l \dot{x}^j,$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{1}{\mathcal{L}} g_{lk} \dddot{x}^l + \frac{1}{\mathcal{L}} \partial_j g_{lk} \dot{x}^l \dot{x}^j - \frac{\dot{\mathcal{L}}}{\mathcal{L}^2} g_{lk} \dot{x}^l.$$
Thus, the Euler-Lagrange equations are

\[ g_{lk} \dddot{x}^l + \partial_j g_{lk} \dot{x}^l \dot{x}^j - \frac{1}{2} \partial_k g_{lj} \dot{x}^l \dot{x}^j = \frac{\dot{L}}{L} g_{lk} \dot{x}^l. \]

Multiplying by \( g^{ki} \), we get

\[ \dddot{x}^i + g^{ki} \partial_j g_{lk} \dot{x}^l \dot{x}^j - \frac{1}{2} g^{ki} \partial_k g_{lj} \dot{x}^l \dot{x}^j = \frac{\dot{L}}{L} \dot{x}^i, \]

and

\[ \dddot{x}^i + \frac{1}{2} g^{ki} (\partial_j g_{lk} + \partial_l g_{jk} - \partial_k g_{lj}) \dot{x}^l \dot{x}^j = \frac{\dot{L}}{L} \dot{x}^i \implies \]

\[ \dddot{x}^i + \Gamma_{lj}^i \dot{x}^l \dot{x}^j = \frac{1}{2} \dot{x}^i \frac{d}{dt} \ln(g_{ij} \dot{x}^i \dot{x}^j). \]

We see that the geodesic equation is retrieved only when the right-hand side is zero, which will occur in general if

\[ g_{ij} \dot{x}^i \dot{x}^j = K = \text{const}. \]

The quantity \( K \) is the square of the length of the tangent vector.

For any given parametrisation of the curve \( C \) with parameter \( t \) one can find a new one with parameter \( s = s(t) \) such that \( K \) is a constant.

To this end one solves the equation

\[ K = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \left( \frac{dt}{ds} \right)^2 \implies \frac{ds}{dt} = \sqrt{\frac{g_{ij} \dot{x}^i \dot{x}^j}{K}} \implies \]

\[ \implies s(t) = \int_a^t \sqrt{\frac{g_{ij} \dot{x}^i \dot{x}^j}{K}} \, d\tau, \]
This is a generalisation of an arc length parametrisation, which is called *affine parametrisation*. For such a parametrisation the length of the tangent vectors remain constant as it is parallel transported along the geodesic.

**Remarks.**

1. If the manifold is Riemannian then $K > 0$, and one can rescale $s$ so that the new $K$ is equal to 1. Then, $s$ is equal to the distance from the point with coordinates $x^i(0)$ to the point with coordinates $x^i(s)$. Tangent vectors to the curve have length 1.

2. If $M$ is Minkowski then $K$ can take any real value, and if $K \neq 0$ one can normalise it to $\pm 1$. The formula (7.7) then is the action of a free *relativistic* particle moving in the Minkowski manifold. If the signature of $M$ is $(-, +, \ldots, +)$, then the curve is called (i) *time-like* if $K = -1$, (ii) *space-like* if $K = +1$, (iii) *light-like* or *null* if $K = 0$.

The corresponding parameters are called *proper time* and *proper distance*.

3. If the manifold is Riemannian then

$$A = \frac{m}{2} \int_a^b g_{ij} \dot{x}^i \dot{x}^j \, dt \quad (7.8)$$

is the action of a free *non-relativistic* particle of mass $m$ moving in the manifold. The Euler-Lagrange equations which follow from (7.8) are the geodesic equations for any choice of $t$ (check it!). The simplest way to derive the geodesic equations is to use (7.8).
8 The General Curvature Tensor

The result of parallel transport of a vector $\xi$ is determined by the equation of parallel transport

$$\frac{dx^k}{dt} \nabla_k \xi^i = \frac{dx^k}{dt} \left( \frac{\partial \xi^i}{\partial x^k} + \Gamma^i_{jk} \xi^j \right) = \frac{d\xi^i}{dt} + \frac{dx^k}{dt} \Gamma^i_{jk} \xi^j = 0.$$ 

If $\Gamma^i_{jk} = 0$ the solution to this equation is trivial.

It would be useful to have a criterion for the existence of coordinates in terms of which $\Gamma^i_{jk}$ vanish. The solution to this problem turns out to be the familiar “equality of mixed partial derivatives”:

$$\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}.$$ 

To show this, let us assume that there are Euclidean coordinates such that $\Gamma^i_{jk} = 0$. Then in those coordinates

$$\nabla_k T^{(i)}_{(j)} = \partial_k T^{(i)}_{(j)},$$

and by “equality of mixed partials”

$$\left( \nabla_k \nabla_l - \nabla_l \nabla_k \right) T^{(i)}_{(j)} = 0.$$ 

Since the lhs of this equation is a tensor the equality will hold in all coordinate systems.
If our connection is arbitrary then
\[ \nabla_i \xi^i = \partial_i \xi^i + \xi^p \Gamma^i_{pl}, \quad \nabla_k \xi_l = \partial_k \xi_l - \xi_p \Gamma^p_{lk}, \]
and therefore
\[ [\nabla_k, \nabla_l] \xi^i = \partial_k (\partial_l \xi^i + \xi^p \Gamma^i_{pl}) + \nabla_l \xi^p \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l) \]
\[ = \partial_k \xi^p \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_l \xi^p \Gamma^i_{pk} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l) \]
\[ = (\nabla_k \xi^p - \xi^q \Gamma^p_{qk}) \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_l \xi^p \Gamma^i_{pk} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l) \]
\[ = -\xi^q \Gamma^p_{qk} \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l) \]
\[ = \left( \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk} \right) \xi^j + \left( \Gamma^j_{kl} - \Gamma^j_{lk} \right) \nabla_j \xi^i. \]

Thus, introducing the torsion tensor
\[ T^j_{kl} = \Gamma^j_{kl} - \Gamma^j_{lk}, \]
and the Riemann curvature tensor
\[ R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk}, \]
one gets the Ricci identity
\[ [\nabla_k, \nabla_l] \xi^i = R^i_{jkl} \xi^j + T^j_{kl} \nabla_j \xi^i. \quad (8.9) \]

For any symmetric connection we have
\[ [\nabla_k, \nabla_l] \xi^i = R^i_{jkl} \xi^j. \quad (8.10) \]
Thus, if the Riemann curvature tensor is not identically zero then the corresponding connection is not Euclidean.
Let us derive coordinate-free formulae for the curvature and torsion tensors. For arbitrary vector fields $\xi, \eta, \zeta$ we set

$$[T(\xi, \eta)]^i = T^i_{kl}\xi^k\eta^l, \quad [R(\xi, \eta)\zeta]^i = R^i_{jkl}\xi^k\eta^l\zeta^j.$$  

**Lemma 30.1.3.** For arbitrary $\xi, \eta, \zeta$, the following equations hold

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta], \quad (8.11)$$

$$R(\xi, \eta)\zeta = [\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi, \eta]} \zeta, \quad (8.12)$$

where $[\xi, \eta]$ is the commutator of the vector fields $\xi, \eta$.

**Proof.** We first show that the rhs of (8.11) and (8.12) are linear in $\xi, \eta, \zeta$. We replace $\xi \rightarrow f\xi$ where $f$ is a function, and use

$$\nabla_{f\xi} = f\nabla_\xi, \quad [f\xi, \eta] = f [\xi, \eta] - (\partial_\eta f)\xi,$$

to get

$$T(f\xi, \eta) = \nabla_{f\xi} \eta - \nabla_\eta (f\xi) - [f\xi, \eta]$$

$$= f(\nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]) - (\nabla_\eta f)\xi + (\partial_\eta f)\xi = f T(\xi, \eta),$$

$$R(f\xi, \eta)\zeta = (\nabla_{f\xi} \nabla_\eta - \nabla_\eta \nabla_{f\xi}) \zeta - \nabla_{[f\xi, \eta]} \zeta$$

$$= f ([\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi, \eta]} \zeta) - \nabla_\eta f \nabla_\xi \zeta + \partial_\eta f \nabla_\xi \zeta = f R(\xi, \eta)\zeta.$$

The linearity in $\eta$ follows similarly.
Finally, $\zeta \to f\zeta$

$$R(\xi, \eta) (f\zeta) = (\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi) (f\zeta) - \nabla_{[\xi,\eta]}(f\zeta)$$

$$= f\left( [\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi,\eta]} \zeta \right)$$

$$+ (\nabla_\eta f)(\nabla_\xi \zeta) + (\nabla_\xi f)(\nabla_\eta \zeta) - (\nabla_\xi f)(\nabla_\eta \zeta) - (\nabla_\eta f)(\nabla_\xi \zeta)$$

$$+ ((\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi) f) \zeta - (\nabla_{[\xi,\eta]} f) \zeta$$

$$= f\left( [\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi,\eta]} \zeta \right)$$

$$+ ((\partial_\xi \partial_\eta - \partial_\eta \partial_\xi) f) \zeta - ((\partial_\xi \partial_\eta - \partial_\eta \partial_\xi) f) \zeta = f R(\xi, \eta) \zeta .$$

where we used $\nabla_{[\xi,\eta]} f = \partial_{[\xi,\eta]} f = (\partial_\xi \partial_\eta - \partial_\eta \partial_\xi) f$.

Since the rhs of (8.11) and (8.12) are linear in $\xi, \eta, \zeta$ it is sufficient to check that (8.11) and (8.12) hold for the basis vector fields

$\xi = e_k, \eta = e_l, \zeta = e_j$ in which case $\xi^i = \delta^i_k, \eta^i = \delta^i_l, \zeta^i = \delta^i_j$.

But for these vector fields (8.11) and (8.12) follow immediately from the definitions of $T^i_{kl}$ and $R^i_{jkl}$, e.g.

$$[T(e_k, e_l)]^i = T^i_{kl} = \Gamma^i_{kl} - \Gamma^i_{lk} = (\nabla_{e_k} e_l - \nabla_{e_l} e_k)^i$$

$$= (\nabla_{e_k} e_l - \nabla_{e_l} e_k - [e_k, e_l])^i$$

and similarly for $[R(e_k, e_l)e_j]^i$. 
9 The symmetries of the Curvature Tensor

Theorem 30.2.1.

1. The tensor $R_{jkl}^i$ is always skew-symmetric in the indices $k$ and $l$

$$R_{jkl}^i + R_{jlk}^i = 0.$$  \hfill (9.13)

2. If the connection is symmetric, then

$$R_{jkl}^i + R_{klj}^i + R_{ijl}^i = 0.$$  \hfill (9.14)

3. If the connection is compatible with the metric $g_{ij}$, and we define

$$R_{ijkl} \equiv g_{ip}R_{jkl}^p,$$

then the tensor $R_{ijkl}$ is skew-symmetric in the indices $i$ and $j$

$$R_{ijkl} + R_{jikl} = 0.$$  \hfill (9.16)

4. If the connection is both symmetric and compatible with the metric $g_{ij}$, then the tensor $R_{ijkl}$ is symmetric under the exchange of the pairs of the indices $ij$ and $kl$

$$R_{ijkl} - R_{klij} = 0.$$  \hfill (9.17)

Proof.

1. is obvious

2. Let $e_i, i = 1, \ldots, n$ be the standard basis of vectors at every point. Since the connection is symmetric one has

$$[\nabla_k, \nabla_l]\xi^i = R_{jkl}^i \xi^j \quad \Rightarrow \quad [\nabla_k, \nabla_l]\xi^i e_i = R_{jkl}^i \xi^j e_i.$$

Choosing $\xi^i e_i = e_j$, i.e. $\xi^i = \delta^i_j$, one gets

$$R_{jkl}^i e_i = [\nabla_k, \nabla_l]e_j.$$
Hence, \( R_{jkl}^i + R_{klj}^i + R_{ijk}^l = 0 \) will follow if we show that

\[
I_{jkl} \equiv [\nabla_k, \nabla_l]e_j + [\nabla_l, \nabla_j]e_k + [\nabla_j, \nabla_k]e_l
\]

vanish. We have

\[
I_{jkl} = \nabla_k(\nabla_l e_j - \nabla_j e_l) + \nabla_l(\nabla_j e_k - \nabla_k e_j) + \nabla_j(\nabla_k e_l - \nabla_l e_k).
\]

Now,

\[
(\nabla_l e_j)^i = \partial_l e_j^i + \Gamma_{pl}^i e_j^p = \Gamma_{jl}^i.
\]

Since \( \Gamma_{jl}^i \) is symmetric \( \nabla_l e_j - \nabla_j e_l = 0 \), and therefore \( I_{jkl} = 0 \).

3. Since \( \Gamma_{kl}^i \) is compatible with \( g_{ij} \), we have \( \nabla_k g_{ij} = 0 \). From

\[
[\nabla_k, \nabla_l]\xi^i = R_{jkl}^i \xi^j + T_{kl}^j \nabla_j \xi^i,
\]

we get (note that the proof in the book is wrong!)

\[
\langle [\nabla_k, \nabla_l]\xi, \xi \rangle = g_{ip} R_{jkl}^i \xi^j \xi^p + g_{ip} T_{kl}^j \nabla_j \xi^i \xi^p
\]

\[
= R_{pjk}^i \xi^j \xi^p + T_{kl}^j \langle \nabla_j \xi, \xi \rangle = R_{pjk}^i \xi^j \xi^p + \frac{1}{2} T_{kl}^j \partial_j \langle \xi, \xi \rangle.
\]

Now, if \( f \) is a scalar, e.g. \( \langle \xi, \xi \rangle \), one has

\[
[\nabla_k, \nabla_l]f = \nabla_k \partial_l f - (k \leftrightarrow l) = -\Gamma_{lk}^j \partial_j f + \Gamma_{kl}^j \partial_j f = T_{kl}^j \partial_j f.
\]

Hence,

\[
\frac{1}{2} T_{kl}^j \partial_j \langle \xi, \xi \rangle = \frac{1}{2} [\nabla_k, \nabla_l] \langle \xi, \xi \rangle = \frac{1}{2} (2 \nabla_k \langle \nabla_l \xi, \xi \rangle - 2 \nabla_l \langle \nabla_k \xi, \xi \rangle)
\]

\[
= \langle [\nabla_k, \nabla_l] \xi, \xi \rangle.
\]

Thus,

\[
R_{pjk}^i \xi^j \xi^p = 0 \quad \implies \quad R_{ijkl} + R_{jikl} = 0.
\]
4. If the connection is both symmetric and compatible with the metric \( g_{ij} \), then we can use the previous three symmetries.

We have from (9.14)

\[
R_{ijkl} + R_{iklj} + R_{iljk} = 0, \quad R_{jikl} + R_{jkli} + R_{jlik} = 0 \quad \implies \\
2R_{ijkl} + R_{iklj} + R_{iljk} - R_{jkli} - R_{jlik} = 0,
\]

(9.18)

\[
R_{kjl} + R_{kil} + R_{klji} = 0, \quad R_{ljki} + R_{lkij} + R_{lij} = 0 \quad \implies \\
2R_{klij} - R_{kjil} - R_{kilj} - R_{ljki} - R_{lijk} = 0.
\]

(9.19)

Subtracting (9.19) from (9.18), and using (9.13) and (9.16), we get

\[
R_{ijkl} - R_{klij} = 0.
\]
10 Normal Coordinates

Given \( p \in M \) and a chart \((x^i)\) we can find a new chart \((\hat{x}^i)\) such that

\[
\hat{\Gamma}^i_{(jk)}(p) = \frac{1}{2}(\hat{\Gamma}^i_{jk}(p) + \hat{\Gamma}^i_{kj}(p)) = 0 ,
\]
or equivalently \( \hat{\Gamma}^i_{jk}(p) = \frac{1}{2}T^i_{jk}(p) \).

Let \( p \) have coordinates \( x^i = 0, \hat{x}^i = 0, i = 1, \ldots, n \).

Let us look for \((\hat{x}^i)\) of the form

\[
\hat{x}^i = x^i + \frac{1}{2}Q^i_{jk}x^jx^k , \tag{10.20}
\]

where \( Q^i_{jk} = Q^i_{kj} \) are some constants to be found from \( \hat{\Gamma}^i_{(jk)}(p) = 0 \).

For small \( |x| = \max_i |x^i| \) we can invert (10.20)

\[
x^i = \hat{x}^i - \frac{1}{2}Q^i_{jk}\hat{x}^j\hat{x}^k + \mathcal{O}(|x|^3) , \tag{10.21}
\]

Then

\[
\frac{\partial \hat{x}^i}{\partial x^j} = \delta^i_j + Q^i_{jk}x^k , \quad \frac{\partial \hat{x}^i}{\partial x^j} \big|_p = \delta^i_j ,
\]

\[
\frac{\partial x^i}{\partial \hat{x}^j} = \delta^i_j - Q^i_{jk}\hat{x}^k + \mathcal{O}(|x|^2) , \quad \frac{\partial x^i}{\partial \hat{x}^j} \big|_p = \delta^i_j ,
\]

\[
\frac{\partial^2 x^i}{\partial \hat{x}^j \partial \hat{x}^k} = -Q^i_{jk} + \mathcal{O}(|x|) , \quad \frac{\partial^2 x^i}{\partial \hat{x}^j \partial \hat{x}^k} \big|_p = -Q^i_{jk} . \tag{10.22}
\]

The connection transforms as

\[
\hat{\Gamma}^i_{jk} = \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \hat{x}^j} \frac{\partial x^c}{\partial \hat{x}^k} \Gamma^a_{bc} + \frac{\partial^2 x^a}{\partial \hat{x}^j \partial \hat{x}^k} .
\]
Thus, at \( p \) we have
\[
\hat{\Gamma}_{jk}^i(p) = \Gamma_{jk}^i(p) - Q_{jk}^i.
\]
So, if we choose \( Q_{jk}^i = \Gamma_{(jk)}^i(p) \), then \( \hat{\Gamma}_{(jk)}^i(p) = 0 \).

Let the coordinates \( (x^i) \) be normal at \( p \) where \( p \) has coordinates \( x^i = 0 \), \( i = 1, \ldots, n \). If the connection is symmetric, then at \( p \) we have
\[
R_{jkl}^i(p) = \partial_k \Gamma_{jl}^i(p) - \partial_l \Gamma_{jk}^i(p),
\]
and
\[
\nabla_m R_{jkl}^i(p) = \partial_m R_{jkl}^i(p) = \partial_m \partial_k \Gamma_{jl}^i(p) - \partial_m \partial_l \Gamma_{jk}^i(p),
\]
This can be used to prove the **Bianchi identities**
\[
\nabla_m R_{ikl}^n + \nabla_i R_{him}^n + \nabla_k R_{ilm}^n = 0.
\]
Riemann normal coordinates.

The choice of normal coordinates is not unique, and if the connection is symmetric, then given \( p \in M \) and a chart with normal coordinates \((x^i)\) we can find a new chart with normal coordinates \((\hat{x}^i)\) such that not only \( \Gamma^i_{jk}(p) = \hat{\Gamma}^i_{jk}(p) = 0 \) but in addition

\[
\hat{\partial}_l \hat{\Gamma}^i_{jk}(p) + \hat{\partial}_j \hat{\Gamma}^i_{kl}(p) + \hat{\partial}_k \hat{\Gamma}^i_{lj}(p) = 0,
\]

(10.23)

where \( \hat{\partial}_l \equiv \frac{\partial}{\partial \hat{x}^l} \). Note that the expression on the lhs is symmetric under the exchange of the indices \( j, k, l \) because the connection is symmetric.

The relation (10.23) allows one to derive the formula (Check this!)

\[
\hat{\partial}_k \hat{\Gamma}^i_{jl}(p) = \frac{1}{3} (\hat{R}^i_{jkl}(p) + \hat{R}^i_{ljk}(p)).
\]

If the connection is symmetric and compatible with the metric, then \( \hat{\partial}_k \hat{g}_{ij}(p) = 0 \) and the metric has the expansion (Check this!)

\[
\hat{g}_{ij}(p) = g_{ij}(p) + \frac{1}{3} \hat{R}_{iklj}(p) \hat{x}^k \hat{x}^l + O(|x|^3).
\]

\( g_{ij}(p) \) can be diagonalised by an orthogonal transformation of \((\hat{x}^i)\): \( \hat{x}^i = A^i_j y^j \). Then by rescaling \( y^i: y^i = \lambda_i z^i \), one can get the expansion

\[
g_{ij}(z) = \delta_{ij} + \frac{1}{3} R_{iklj}(0) z^k z^l + O(|z|^3),
\]

where \( R_{iklj}(0) \) is the Riemann curvature tensor in \((z^i)\) chart.
To prove (10.23) let \( p \) have coordinates \( x^i = 0, \hat{x}^i = 0, i = 1, \ldots, n \).

Let us look for \((\hat{x}^i)\) of the form

\[
\hat{x}^i = x^i + \frac{1}{6} Q^i_{jkl} x^j x^k x^l, \tag{10.24}
\]

where \( Q^i_{jkl} \) are some constants symmetric under the exchange of the indices \( j, k, l \). They are to be found from (10.23).

For small \(|x| = \max_i |x^i|\) we can invert (10.24)

\[
x^i = \hat{x}^i - \frac{1}{6} Q^i_{jkl} \hat{x}^j \hat{x}^k \hat{x}^l + \mathcal{O}(|x|^4), \tag{10.25}
\]

Then

\[
\frac{\partial \hat{x}^i}{\partial x^j} = \delta^i_j + \frac{1}{2} Q^i_{jkl} x^k x^l, \quad \frac{\partial \hat{x}^i}{\partial x^j} \bigg|_p = \delta^i_j, \\
\frac{\partial x^i}{\partial \hat{x}^j} = \delta^i_j - \frac{1}{2} Q^i_{jkl} \hat{x}^k \hat{x}^l + \mathcal{O}(|x|^3), \quad \frac{\partial x^i}{\partial \hat{x}^j} \bigg|_p = \delta^i_j, \\
\frac{\partial^2 x^i}{\partial \hat{x}^j \partial x^k} = -Q^i_{jkl} \hat{x}^l + \mathcal{O}(|x|^2), \quad \frac{\partial^2 x^i}{\partial \hat{x}^j \partial x^k} \bigg|_p = 0, \\
\frac{\partial^3 x^i}{\partial \hat{x}^j \partial \hat{x}^k \partial x^l} = -Q^i_{jkl} + \mathcal{O}(|x|), \quad \frac{\partial^3 x^i}{\partial \hat{x}^j \partial \hat{x}^k \partial x^l} \bigg|_p = -Q^i_{jkl}. \tag{10.26}
\]

The derivative of the connection transforms as

\[
\hat{\Gamma}^i_{jk} = \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \hat{x}^j} \frac{\partial x^c}{\partial \hat{x}^k} \frac{\partial x^d}{\partial \hat{x}^l} \partial_a \Gamma^a_{bc} + \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial^3 x^a}{\partial \hat{x}^j \partial \hat{x}^k \partial \hat{x}^l} + \left[ \frac{\partial}{\partial \hat{x}^l} \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \hat{x}^j} \frac{\partial x^c}{\partial \hat{x}^k} \right] \Gamma^a_{bc} + \left[ \frac{\partial}{\partial \hat{x}^l} \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial^3 x^a}{\partial \hat{x}^j \partial x^b} \right].
\]

Thus, at \( p \) we have

\[
\hat{\Gamma}^i_{jk}(p) = \partial_i \Gamma^i_{jk}(p) - Q^i_{jkl}.
\]

So, if we choose \( Q^i_{jk} = \frac{1}{3} \partial_i \Gamma^i_{jk}(p) + \partial_j \Gamma^i_{kl}(p) + \partial_k \Gamma^i_{lj}(p) \),

then \( \hat{\Gamma}^i_{jk}(p) + \hat{\Gamma}^i_{kl}(p) + \hat{\Gamma}^i_{lj}(p) = 0 \).
11 Ricci Tensor, Scalar Curvature, Einstein Tensor and Weyl Tensor

Def 30.3.1. The trace (or contraction)

$$R_{jl} = R^i_{jil}$$

of the Riemann curvature tensor is called the Ricci tensor. Explicitly it is given by

$$R_{jl} = R^i_{jil} = \partial_i \Gamma^i_{jl} - \partial_l \Gamma^i_{ji} + \Gamma^i_{pi} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{ji}. $$

If the connection is symmetric and compatible with the metric, then

$$R_{jl} = R_{lj},$$

because

$$\partial_l \Gamma^i_{ji} = \partial_l \left( \frac{1}{2g} \partial_j g \right) = \frac{1}{2} \partial_l \partial_j \ln |g|,$$

and all the other terms are obviously symmetric too.

Def 30.3.2. The scalar

$$R = g^{jl} R_{jl} = g^{jl} R^i_{jil}$$

is called the scalar curvature of the underlying Riemann manifold with the metric tensor $g_{ij}$.
**Def 3.** If the connection is symmetric and compatible with the metric, then the tensor

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

is called the *Einstein tensor*. Note that \((\dim M = n)\)

$$g^{ij} G_{ij} = R - \frac{1}{2} R n = \frac{2 - n}{2} R .$$

It appears in Einstein’s equations

$$G_{ij} + \Lambda g_{ij} = \frac{8 \pi G}{c^4} T_{ij} ,$$

where \(\Lambda\) is the cosmological constant,
\(G\) is Newton’s gravitational constant, \(c\) is the speed of light in vacuum, and \(T_{ij}\) is the stress-energy (or energy-momentum) tensor of matter.

In the vacuum \(T_{ij} = 0\), and one gets

$$G_{ij} + \Lambda g_{ij} = 0 \implies \frac{2 - n}{2} R + \Lambda n = 0 \implies R = \frac{2n}{n - 2} \Lambda = \text{const} ,$$

and

$$R_{ij} = \frac{2}{n - 2} \Lambda g_{ij} .$$

If the metric satisfies this equation, then the manifold is called the *Einstein manifold*. 

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Def 4. The tensor

\[ C_{ijkl} = R_{ijkl} - \frac{2}{n-2} (g_{i[k} R_{l]j} - g_{j[k} R_{l]i}) + \frac{2}{(n-2)(n-1)} g_{i[k} g_{l]j} R \]

is called the Weyl tensor. Here \([kl]\) means anti-symmetrisation with respect to \(k\) and \(l\), e.g. \(g_{i[k} R_{l]j} = \frac{1}{2} (g_{ik} R_{lj} - g_{il} R_{kj})\).

All contractions over the Weyl tensor vanish (check it!).

The Weyl tensor is sometimes called the conformal tensor because it is invariant under conformal transformations

\[ g_{ij} \rightarrow \lambda g_{ij} , \quad C_{ijkl}^{i} \rightarrow C_{ijkl}^{i} . \]

If \( C_{ijkl} = 0 \) and \( \dim M \geq 4 \), then the manifold is locally conformally flat.
12 The Curvature Tensor in 2 dimensions

In 2 dimensions the curvature tensor is given by

\[ R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}) , \]

In 2D all nonvanishing components of \( R_{ijkl} \) are expressed through \( R_{1212} \). Then the components \( g^{ij} \) are expressed through \( g_{ij} \) as

\[ g^{11} = \frac{g_{22}}{g}, \quad g^{22} = \frac{g_{11}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \]

\[ g = g_{11} g_{22} - g_{12} g_{21}, \quad \frac{1}{g} = g^{11} g^{22} - g^{12} g^{21}. \]

Using these formulae, one finds

\[ R_{11} = g^{pq} R_{q1p1} = g^{22} R_{2121} = g^{22} R_{1212}, \]

\[ R_{12} = g^{pq} R_{q1p2} = g^{12} R_{2112} = -g^{12} R_{1212}, \]

\[ R_{22} = g^{pq} R_{q2p2} = g^{11} R_{1212}, \]

\[ R = g^{pq} R_{qp} = g^{11} R_{11} + 2g^{12} R_{12} + g^{22} R_{22} \]

\[ = 2(g^{11} g^{22} - g^{12} g^{21}) R_{1212} = \frac{2}{g} R_{1212}. \]

So,

\[ R_{1212} = \frac{g}{2} R = \frac{1}{2} (g_{11} g_{22} - g_{12} g_{21}) R \Rightarrow R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}) \Rightarrow \]

\[ R_{ab} = \frac{1}{2} R g_{ab} \Rightarrow G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 0. \]
Another (more elegant) way to derive the formula is to notice that the combination \(g_{ac}g_{bd} - g_{ad}g_{bc}\) is a tensor which has the same symmetries as \(R_{abcd}\) (Check this!). Thus
\[
R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}),
\]
where \(K\) is a scalar. Computing \(R_{ab}\) one gets
\[
R_{ab} = Kg_{ab} \quad \Rightarrow \quad R = 2K.
\]

### 13 The Curvature Tensor in 3 dimensions

In 3D, the curvature tensor is given by
\[
R_{abcd} = R_{ac}g_{bd} - R_{ad}g_{bc} + g_{ac}R_{bd} - g_{ad}R_{bc} - \frac{1}{2} R(g_{ac}g_{bd} - g_{ad}g_{bc}).
\] (13.27)

In 3D all non vanishing components of \(R_{ijkl}\) are expressed through
\[
R_{1212}, \quad R_{1213}, \quad R_{1223}, \quad R_{1313}, \quad R_{1323}, \quad R_{2323}.
\]

The Ricci tensor is symmetric and has six independent components. The definition \(R_{bd} = g^{ac}R_{abcd}\) can be considered as a set of linear equations on the Riemann tensor components.

Since the number of independent components of \(R_{ab}\) and \(R_{acbd}\) is the same, the linear equations must have a unique solution, and therefore the Riemann tensor components \(R_{abcd}\) are linear combinations of \(R_{ab}\). Thus, all one has to do is to check that the formula (13.27) is consistent with \(R_{bd} = g^{ac}R_{abcd}\). This is indeed so
\[
g^{ac}R_{abcd} = Rg_{bd} - R_{bd} + 3R_{bd} - R_{bd} - \frac{1}{2} R(3g_{bd} - g_{bd}) = R_{bd}.
\]
14 The Curvature Tensor in $n \geq 4$ dimensions

Let the connection be torsion-free and compatible with the metric. How many linearly independent components does $R_{ijkl}$ have in arbitrary dimensions?

Recall that $R_{ijkl}$ satisfies the following algebraic relations

$$R_{ijkl} + R_{ijlk} = 0, \quad R_{ijkl} + R_{jikl} = 0,$$
$$R_{ijkl} - R_{klij} = 0, \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$ 

Thanks to $R_{ijkl} = R_{klij}$ one can think about $R_{ijkl}$ as a symmetric matrix $R_{AB} = R_{BA}$ where $A = ij$ and $B = kl$ are multi-indices. Since $R_{ijkl} = -R_{ijlk}, R_{ijkl} = -R_{jikl}$ the indices $A$ and $B$ take $n(n-1)/2$ values. Thus, a generic symmetric matrix $R_{AB}$ would have

$$\frac{1}{2} n(n-1) \left( \frac{n(n-1)}{2} + 1 \right) = \frac{n(n-1)(n^2 - n + 2)}{8}$$

independent components.

One has to subtract from this number the number of independent relations due to $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. We note that

$$R_{i[jkl]} \equiv R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$

and it is skew-symmetric under permutations of $j, k, l$. 

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Moreover,

\[ R_{ijkl} \equiv R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{jlkl} - R_{jlik} - R_{jkli} = -R_{j[kli]} , \]

and similarly

\[ R_{i[jkl]} = R_{k[lji]} = -R_{l[ijk]} . \]

Now consider the skew-symmetric part of \( R_{ijkl} \)

\[ R_{[ijkl]} \equiv R_{ijkl} - R_{j[ikl]} - R_{k[jil]} - R_{l[jki]} = 4R_{i[jkl]} = 0 . \]

Thus, the relation \( R_{ijkl} + R_{iklj} + R_{iljk} = 0 \) together with the first three relations is equivalent to vanishing of the skew-symmetric part of \( R_{ijkl} \).

Since a fourth-rank skew-symmetric tensor has

\[ n(n - 1)(n - 2)(n - 3)/4! \]

independent components we finally get

\[ \frac{n(n - 1)(n^2 - n + 2)}{8} - \frac{n(n - 1)(n^2 - 5n + 6)}{24} = \frac{n(n - 1)(3n^2 - 3n - n^2 + 5n)}{24} = \frac{n^2(n^2 - 1)}{12} . \]
Left-Invariant Fields

Let $X = (x^i_k)$ be a fixed $n \times n$ matrix. To each $X$ there corresponds the linear transformation

$$A \mapsto AX, \quad A \in \mathbb{R}^{n^2}$$

of the space $\mathbb{R}^{n^2}$ of $n \times n$ real matrices $A$.

We denote by $L_X$ the linear vector field on $\mathbb{R}^{n^2}$ which at the point $A$ takes the value

$$L_X(A) = AX$$

1. (a) Show that the integral curve of $L_X$ satisfying the initial condition $A(0) = A_0$ is given by

$$A = A_0 \exp(tX).$$

(b) Prove that

$$[L_X, L_Y] = L_{[X,Y]}.$$

c) Let $G$ be a matrix group considered as a smooth surface in $\mathbb{R}^{n^2}$ of all $n \times n$ real matrices $A$, and let $G$ be the tangent space to $G$ at the identity. Recall that if $X \in G$ then the matrices $\exp(tX)$ form a one-parameter subgroup of $G$. Show that for each $X \in G$, the vector field $L_X$ is tangent to the surface $G$; hence its restriction to $G$ is a vector field on $G$.

d) **Definition.** A vector field of the form $L_X$ on a classical group $G$ where $X$ is an element of the Lie algebra $G$ of $G$ is called a **left-invariant field** on the group $G$.

Prove that the left-invariant vector fields on a group $G$ form a Lie algebra isomorphic to the Lie algebra $G$ of $G$. 

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2. Invariant Metrics on a Transformation Group

**Definition.** A Euclidean or pseudo-Euclidean scalar product $\langle \cdot, \cdot \rangle_0$ on a Lie algebra $\mathcal{G}$ is called an *invariant scalar product* if all the operators $\text{ad} \, X$, $X \in \mathcal{G}$, are skew-symmetric with respect to $\langle \cdot, \cdot \rangle_0$, i.e. if

$$\langle \text{ad} \, X(Y), Z \rangle_0 = -\langle Y, \text{ad} \, X(Z) \rangle_0.$$ 

Let $\mathcal{G}$ be a Lie algebra of a transformation group $G$, and suppose an invariant scalar product $\langle \cdot, \cdot \rangle_0$ is given on $\mathcal{G}$. Let us use this scalar product and the left-invariant fields on $G$ to introduce a metric on the surface $G$ itself. Let $A$ be any point (i.e. matrix) in $G$; then every vector tangent to $G$ at $A$ has the form $L_X(A)$ for some unique $X \in \mathcal{G}$. It follows that in setting

$$\langle L_X(A), L_Y(A) \rangle_0 = \langle X, Y \rangle_0$$

for all $X, Y \in \mathcal{G}$ we defined the scalar product of any pair of vectors tangent to $G$ at the arbitrary point $A$, i.e. we defined a metric on $G$. This metric is called the *invariant metric on the group $G$.*

Choosing $X = Y = A^{-1}dA$ where $dA = (dA^i_j)$ is the matrix of differentials of $A^i_j$ one gets the square of the element of length on $G$

$$dl^2 = \langle A^{-1}dA, A^{-1}dA \rangle_0.$$
(a) Show that the Euclidean metric on the space $\mathbb{R}^{n^2}$ of all $n \times n$ real matrices induces an invariant metric on $SO(n, R)$.

(b) Let $g^0_{ij}$ be an invariant scalar product on a Lie algebra $G$ with basis $X_1, \ldots, X_n$. Write

$$[X_i, X_j] = c^k_{ij} X_k, \quad c_{kij} = g^0_{kl} c^l_{ij}.$$ 

Show that the tensor $c_{kij}$ is skew-symmetric.

**Solution:** Since $ad X(Y) = [X, Y]$ one gets

$$\langle ad X_i(X_j), X_k \rangle_0 = \langle [X_i, X_j], X_k \rangle_0 = -\langle X_j, [X_i, X_k] \rangle_0 .$$  

(14.28)

Then, one has

$$\langle [X_i, X_j], X_k \rangle_0 = g^0_{mn} [X_i, X_j]^m X^n_k = g^0_{mn} c^m_{pq} X^p_i X^q_j X^n_k$$

$$= c_{npq} X^p_i X^q_j X^n_k ,$$  

(14.29)

$$-\langle X_j, [X_i, X_k] \rangle_0 = -g^0_{mn} [X_i, X_k]^m X^n_j = -g^0_{mn} c^m_{pq} X^p_i X^q_k X^n_j$$

$$= -c_{qpn} X^p_i X^q_j X^n_k .$$  

(14.30)

Thus one gets

$$c_{npq} = -c_{qpn} \quad \implies \quad c_{nqp} = -c_{qnp} ,$$  

(14.31)

where we used that $c_{ijk}$ is skew-symmetric in $j, k$. 

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(c) An invariant metric on the group $SO(p, q)$ can be obtained as the restriction to this group of the pseudo-Euclidean metric

$$\langle X, Y \rangle = \text{tr}(G X G Y^T),$$

where $G = \text{diag} (\mathbb{I}_p, -\mathbb{I}_q)$ is the matrix of the pseudo-Euclidean metric of type $(p, q)$.

What is the type of the resulting pseudo-Riemannian metric on $SO(p, q)$?

Solution: A matrix $O \in SO(p, q)$ satisfies

$$G O G O^T = \mathbb{I}, \quad (14.32)$$

and therefore a matrix $X \in \mathcal{G}$ from the Lie algebra $so(p, q)$ satisfies

$$X^T = -G X G \quad \implies \quad x_{ji} = -g_i g_j x_{ij}. \quad (14.33)$$

Thus,

$$\langle X, X \rangle = \text{tr}(G X G X^T) = -\text{tr}(X^2) = -x_{ij} x_{ji} = 2 \sum_{i<j} g_i g_j x_{ij}^2, \quad (14.34)$$

where $g_i$ are the diagonal entries of $G$, and we took into account that $x_{ij}$ with $i < j$ are independent and provide local coordinates in a neighbourhood of the identity element of $SO(p, q)$. The dimension of $SO(p, q)$ therefore is

$$\dim SO(p, q) = (p + q)(p + q - 1)/2$$

It is clear that the negative terms in $(14.34)$ are obtained for $i = 1, 2, \ldots, p$ and $j = p + 1, \ldots p + q$. Thus, the pseudo-Riemannian metric on $SO(p, q)$ is of type $(\dim SO(p, q) - pq, pq)$.
3. Define a right-invariant field on a matrix group $G$ to be the restriction to $G$ of a vector field of the form $R_X(A) = -XA$. Prove that

$$[R_X, R_Y] = R_{[X,Y]}, \quad [L_X, R_Y] = 0.$$ 

Solution: We have for any $A$

$$[R_X, R_Y](A) = R_X(-YA) - R_Y(-XA) = XYA - YXA = R_{[X,Y]}(A), \quad (14.35)$$

$$[L_X, R_Y](A) = L_X(-YA) - R_Y(-AX) = YAX - YAX = 0. \quad (14.36)$$