

Vector-Valued Functions

1 Parametric curves

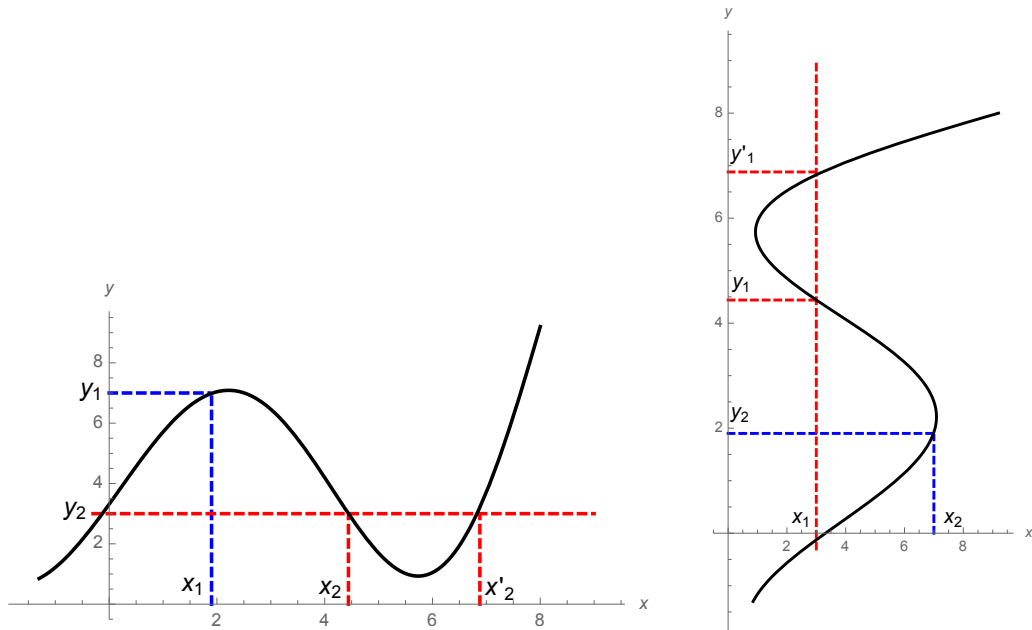


Figure 1: Which curve is a graph of a function?

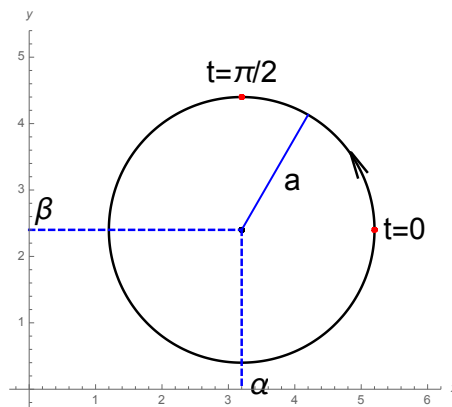


Figure 2: Is it a graph of a function?

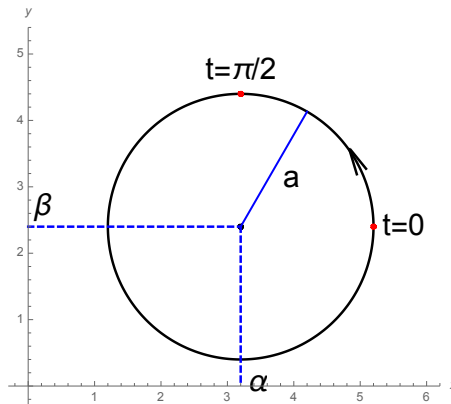


Figure 3: This curve is *not* a graph of a function.

This curve is a circle of radius a centred at (α, β)

$$(x - \alpha)^2 + (y - \beta)^2 = a^2$$

It is a **parametric curve** which can be represented by the eqs

$$x = \alpha + a \cos t, \quad y = \beta + a \sin t, \quad 0 \leq t \leq 2\pi.$$

In general in 2-space a parametric curve is

$$x = f(t), \quad y = g(t), \quad t_0 \leq t \leq t_1.$$

The graph of a function $y = f(x)$ is a parametric curve

$$x = t, \quad y = f(t).$$

The graph of a function $x = f(y)$ is a a parametric curve

$$x = f(t), \quad y = t.$$

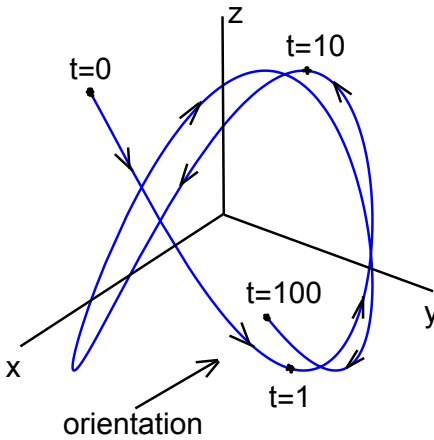


Figure 4: A parametric curve in 3-space

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t_0 \leq t \leq t_1.$$

These parametric eqs generate a curve in 3-space called the **parametric curve** represented by the eqs, and as t increases the point moves in a specific direction which defines the curve **orientation**.

A parametric curve can be considered as a path of a point particle with the parameter t identified with time.

A parametric curve in n -space is represented by n eqs

$$x_1 = f_1(t), \quad x_2 = f_2(t), \quad \dots, \quad x_n = f_n(t), \quad t_0 \leq t \leq t_1.$$

Example 1.

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1.$$

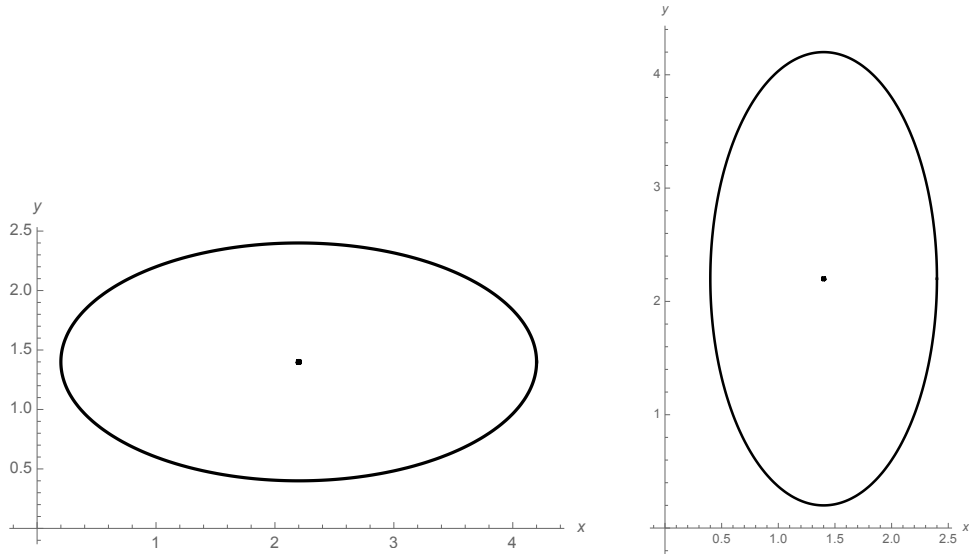


Figure 5: These are ellipses.

This ellipse can be represented by the eqs

$$x = \alpha + a \cos t, \quad y = \beta + b \sin t, \quad 0 \leq t \leq 2\pi.$$

Example 1. Ellipse

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1.$$

Example 2.

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 1.$$

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = -1.$$

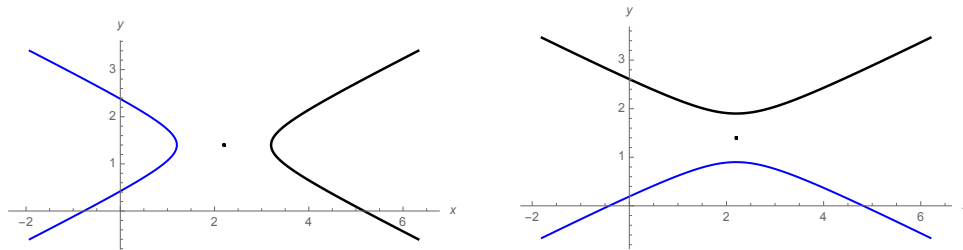


Figure 6: These are hyperbolas.

The right (black) branch of the first hyperbola can be represented by the eqs

$$x = \alpha + a \cosh t, \quad y = \beta + b \sinh t, \quad -\infty < t < \infty.$$

because

$$\cosh t = \frac{e^t + e^{-t}}{2} > 0, \quad \sinh t = \frac{e^t - e^{-t}}{2}, \quad \cosh^2 t - \sinh^2 t = 1.$$

HW: Find a representation of the left (blue) branch of the first hyperbola, and similar representations of both branches of the second hyperbola.

Example 1. Ellipse

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1.$$

Example 2. Hyperbolas

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 1.$$

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = -1.$$

Example 3.

$$y = ax^2.$$

$$y^2 = ax.$$

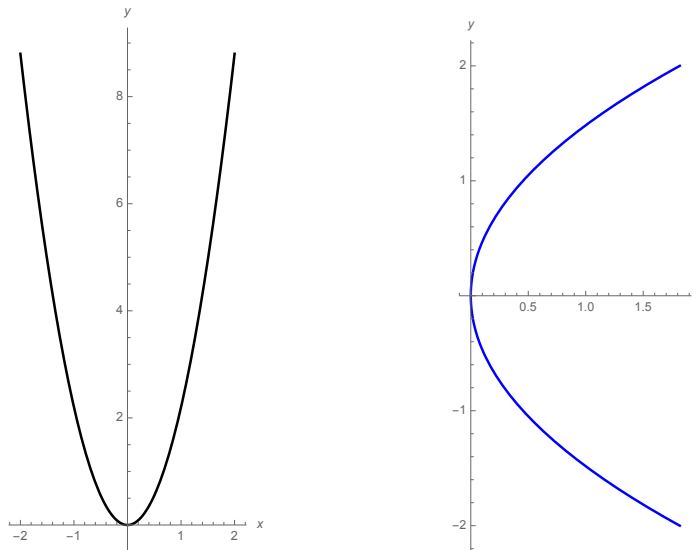


Figure 7: These are parabolas.

They can be represented by the eqs

$$x = t, \quad y = at^2, \quad x = \frac{1}{a}t^2, \quad y = t, \quad -\infty < t < \infty.$$

Ellipses, hyperbolas and parabolas are **conic sections**.

Example 4.

$$\begin{aligned} x &= a \cos t, & y &= a \sin t, & z &= vt, & -\infty < t < \infty. \\ x &= a \sin t, & y &= a \cos t, & z &= vt, & -\infty < t < \infty. \end{aligned}$$

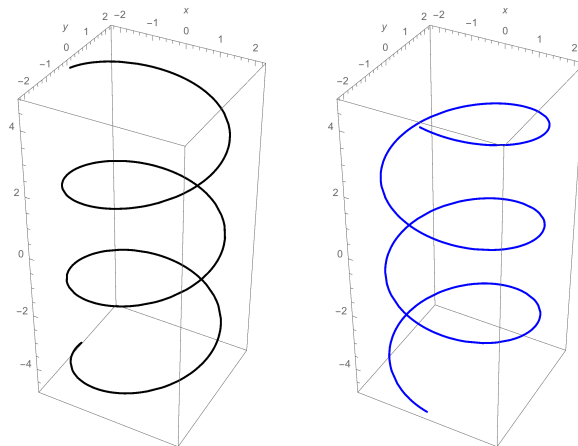


Figure 8: These are circular helices.

Example 5. Line through $\vec{r}_0 = (x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = (v_x, v_y, v_z)$

$$x = x_0 + v_x t, \quad y = y_0 + v_y t, \quad z = z_0 + v_z t, \quad -\infty < t < \infty.$$

Example 5b. Find parametric equations for the line through $(2, 0, 0)$ and $(0, 4, 3)$.

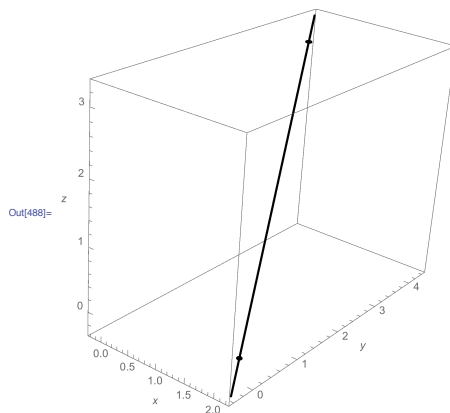


Figure 9: This is the line through $(2, 0, 0)$ and $(0, 4, 3)$.

Let $t = 0$ correspond to $(2, 0, 0)$, and let $t = 1$ corresponds to $(0, 4, 3)$. Then,

$$\begin{aligned} x(0) = x_0 = 2, & \quad x(1) = x_0 + v_x = 0 & \implies & \quad v_x = -2, \\ y(0) = y_0 = 0, & \quad y(1) = y_0 + v_y = 4 & \implies & \quad v_y = 4, \\ z(0) = z_0 = 0, & \quad z(1) = z_0 + v_z = 3 & \implies & \quad v_z = 3. \end{aligned}$$

Thus

$$x = 2 - 2t, \quad y = 4t, \quad z = 3t, \quad -\infty < t < \infty.$$

Example 5. Line through $\vec{r}_0 = (x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = (v_x, v_y, v_z)$

$$x = x_0 + v_x t, \quad y = y_0 + v_y t, \quad z = z_0 + v_z t, \quad -\infty < t < \infty.$$

Example 6. The same line as in Example 5

$$x = x_0 - \frac{1}{2}v_x t, \quad y = y_0 - \frac{1}{2}v_y t, \quad z = z_0 - \frac{1}{2}v_z t, \quad -\infty < t < \infty.$$

Example 7. The same line as in Examples 5 and 6

$$x = x_0 - \frac{1}{2}v_x(t - t_0), \quad y = y_0 - \frac{1}{2}v_y(t - t_0), \quad z = z_0 - \frac{1}{2}v_z(t - t_0).$$

Example 8. This is a segment of the same line as in Examples 5, 6 and 7

$$x = x_0 + v_x t^3, \quad y = y_0 + v_y t^3, \quad z = z_0 + v_z t^3, \quad -1 \leq t \leq 1.$$

Example 9. This is a segment of the same line as in Examples 5, 6 and 7 **but it is run twice!**

$$x = x_0 + v_x(t - 1)^2, \quad y = y_0 + v_y(t - 1)^2, \quad z = z_0 + v_z(t - 1)^2, \quad 0 \leq t \leq 2.$$

$\exists \infty$ many parametric eqs representing the same curve.

Make a change $t = \gamma(\tau)$ where $\gamma(\tau)$ is a one-to-one function on the interval $\tau_0 \leq \tau \leq \tau_1$ of interest.

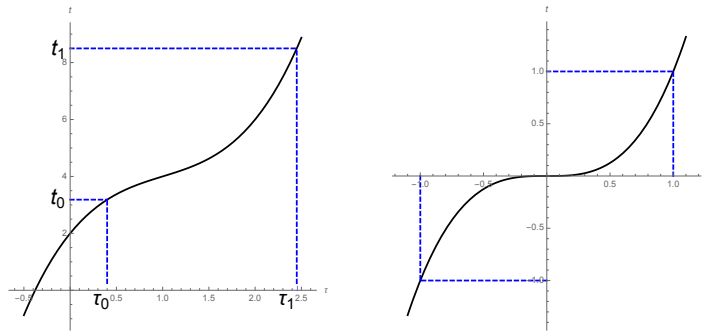


Figure 10: $\gamma(\tau_0) = t_0, \gamma(\tau_1) = t_1;$ $t = \tau^3;$

The parameterisations are considered as equivalent if $\gamma'(\tau) \neq 0$ for $\tau_0 \leq \tau \leq \tau_1$. If $\gamma'(\tau) > 0$ the curves have the same orientation.

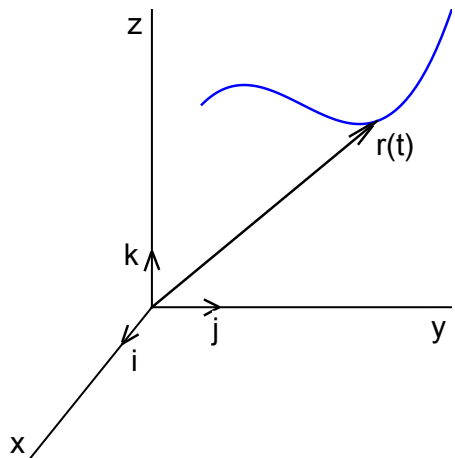
Example.

$$x = \cos t, \quad y = \sin t,$$

$$t = -\tau \implies x = \cos \tau, \quad y = -\sin \tau,$$

$$t = \tau - \frac{\pi}{2} \implies x = \sin \tau, \quad y = -\cos \tau.$$

2 Vector-Valued Functions



$$x = f(t), y = g(t), z = h(t)$$

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

Thus, \vec{r} is a function of t :

$$\vec{r}' = \vec{r}'(t).$$

This is a **vector-valued** function of a real variable t .

$x(t), y(t), z(t)$ are **components** of $\vec{r}(t)$.

The graph of $\vec{r}(t)$ is the parametric curve \mathcal{C} described by the components of $\vec{r}(t)$.

$\vec{r}(t)$ is called the **radius vector** for the curve \mathcal{C} .

The domain of $\vec{r}(t)$ is the intersection of domains of $x(t), y(t), z(t)$.

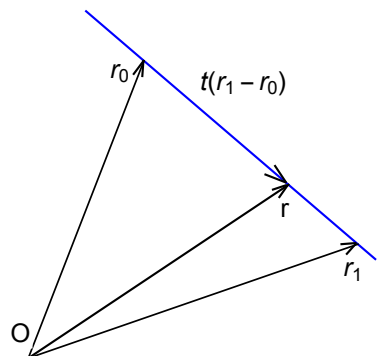
It is called the **natural domain** of $\vec{r}(t)$.

Example.

$$\vec{r}(t) = \sqrt{t-1}\vec{i} + \ln(3-t)\vec{j} + \frac{1}{t-2}\vec{k}.$$

$$D(\vec{r}) = [1, 2) \cup (2, 3).$$

3 Vector form of a line segment



If \vec{r}_0 is a vector with its initial point at O

then the line through the terminal point

\vec{r}_0 and parallel to \vec{v} is

$$\vec{r} = \vec{r}(t) = \vec{r}_0 + t\vec{v}.$$

So, $\vec{r}(0) = \vec{r}_0$, $\vec{r}(1) = \vec{r}_1 = \vec{r}_0 + \vec{v}$

$$\Rightarrow \vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1-t)\vec{r}_0 + t\vec{r}_1$$

This is a **two-point form** of a line.

If $0 \leq t \leq 1$, then $\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1$ represents the line segment from \vec{r}_0 to \vec{r}_1 .

4 Calculus of Vector-valued Functions

$$\lim_{t \rightarrow a} \vec{r}(t) = \left(\lim_{t \rightarrow a} x(t) \right) \vec{i} + \left(\lim_{t \rightarrow a} y(t) \right) \vec{j} + \left(\lim_{t \rightarrow a} z(t) \right) \vec{k},$$

where all the three limits must exist.

A vector-valued function is continuous at $t = a$ (or on an interval I) if all its components are. Then

$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a) \quad \forall a \in I.$$

Derivative of \vec{r} with respect to t is the following vector

$$\vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$

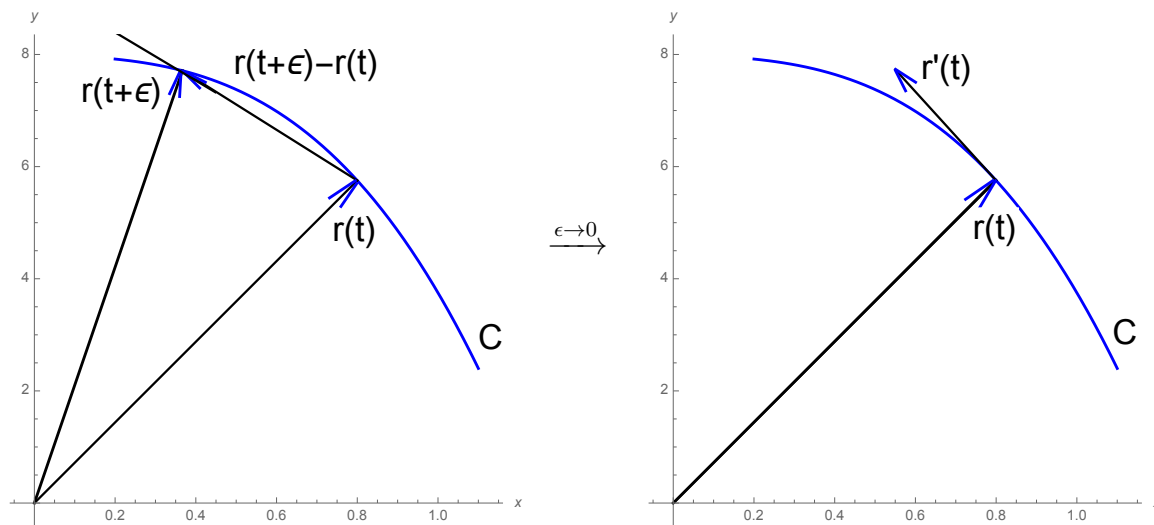
$\vec{r}(t)$ is differentiable if all its components are.

Notations

$$\frac{d}{dt} \vec{r}(t), \quad \frac{d\vec{r}}{dt}, \quad \vec{r}'(t), \quad \dot{\vec{r}}, \quad \dot{\vec{r}}(t), \quad \dot{\vec{r}}, \quad \frac{d}{dt}[\vec{r}(t)].$$

Derivative Rules

1. $\frac{d}{dt} [a \vec{r}_1(t) + b \vec{r}_2(t)] = a \vec{r}_1'(t) + b \vec{r}_2'(t)$, a, b are const
2. $\frac{d}{dt} [f(t) \vec{r}(t)] = f'(t) \vec{r}(t) + f(t) \vec{r}'(t)$,



If $\vec{r}'(t) \neq 0$ and it is positioned with its initial point at the terminal point of \vec{r} then \vec{r}' is tangent to \mathcal{C} and points in the direction of increasing parameter.

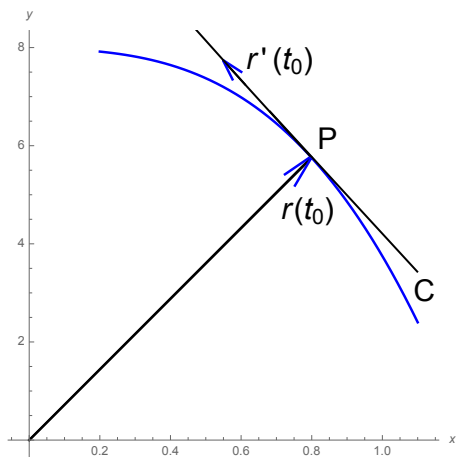
In mechanics it is velocity of a particle moving along \mathcal{C} .

Example.

$$\vec{r}(t) = 2 \cos \frac{\pi}{2} t \vec{i} + \sqrt[3]{1 + 3e^{2t}} \vec{j} + \int_1^{\ln t} \frac{e^{3u}}{u} du \vec{k}.$$

$$\vec{r}'(t) = ?$$

5 Tangent lines to graphs of vector-valued functions



Let P be a point on the graph \mathcal{C} of a VV function $\vec{r}(t)$, and let $\vec{r}'(t_0) \neq 0$ where $\vec{r}(t_0)$ is the radius vector from O to P .

Then, $\vec{r}'(t_0)$ is a **tangent vector** to \mathcal{C} at P , and the line through P that is parallel to $\vec{r}'(t_0)$ is the tangent line to \mathcal{C} at P .

The tangent line is given by the vector eq

$$\vec{R}(t) = \vec{r}_0 + t \vec{v}_0, \quad \vec{v}_0 = \vec{r}'(t_0), \quad \vec{r}_0 = \vec{r}(t_0).$$

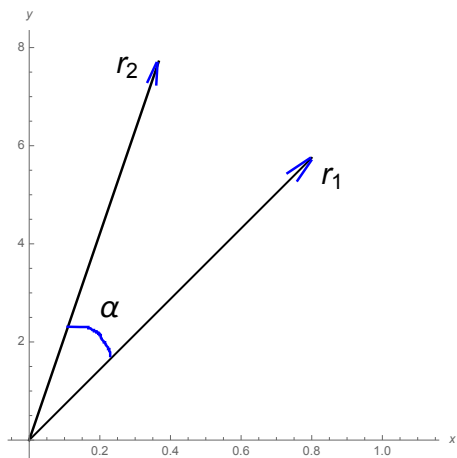
The unit tangent vector is $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{r}'}{|\vec{r}'|}$.

Example.

$$\vec{r}(t) = a \sin t \vec{i} + a \cos t \vec{j} + vt \vec{k},$$

$$\vec{r}'(t) = ?, \quad |\vec{r}'(t)| = ?, \quad \vec{T}(t) = ?, \quad \vec{R}(t) \text{ with } t_0 = \frac{\pi}{3}?,$$

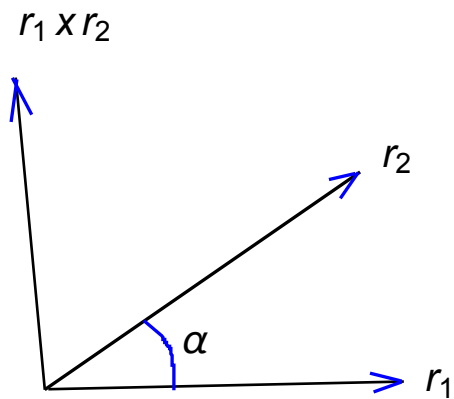
6 Derivatives of dot and cross products



$$\vec{r}_1 = (x_1, y_1, z_1), \quad \vec{r}_2 = (x_2, y_2, z_2).$$

$$\vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$= |\vec{r}_1| |\vec{r}_2| \cos \alpha = \vec{r}_2 \cdot \vec{r}_1$$



$$\begin{aligned} \vec{r}_1 \times \vec{r}_2 &= +(y_1 z_2 - y_2 z_1) \vec{i} \\ &\quad -(x_1 z_2 - x_2 z_1) \vec{j} \\ &\quad + (x_1 y_2 - x_2 y_1) \vec{k} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\vec{r}_2 \times \vec{r}_1 \end{aligned}$$

$$|\vec{r}_1 \times \vec{r}_2| = |\vec{r}_1| |\vec{r}_2| \sin \alpha,$$

$$\vec{r}_1 \times \vec{r}_2 \perp \vec{r}_1 \text{ and } \vec{r}_2.$$

$$\frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 + \vec{r}_1 \cdot \frac{d\vec{r}_2}{dt},$$

$$\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \frac{d\vec{r}_1}{dt} \times \vec{r}_2 + \vec{r}_1 \times \frac{d\vec{r}_2}{dt}.$$

Theorem. If $|\vec{r}(t)|$ is a constant then $\vec{r}(t) \cdot \vec{r}'(t) = 0$.

Proof:

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d}{dt}|\vec{r}|^2 = 0,$$

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 2\vec{r} \cdot \frac{d\vec{r}}{dt},$$

Thus, $\vec{r}(t) \perp \vec{r}'(t)$, and $\vec{T}(t) \perp \vec{T}'(t)$.

Example. A curve on the surface of a sphere that is centred at the origin has $|\vec{r}(t)| = \text{const}$, thus $\vec{r}(t) \perp \vec{r}'(t)$.

7 Integrals of Vector-valued Functions

$$\int_a^b \vec{r}(t) dt = \left(\int_a^b x(t) dt \right) \vec{i} + \left(\int_a^b y(t) dt \right) \vec{j} + \left(\int_a^b z(t) dt \right) \vec{k}$$

Example 1.

$$\int_0^1 (\sqrt{3t+1} \vec{i} + e^{2t} \vec{j} + 3 \sin(\pi t) \vec{k}) dt = ?$$

Rules of integration

$$\int_a^b (c_1 \vec{r}_1(t) + c_2 \vec{r}_2(t)) dt = c_1 \int_a^b \vec{r}_1(t) dt + c_2 \int_a^b \vec{r}_2(t) dt$$

An **antiderivative** for a v-v function $\vec{r}(t)$ is a v-v function $\vec{R}(t)$ such that

$$\vec{R}'(t) = \vec{r}(t) \implies \int \vec{r}(t) dt = \vec{R}(t) + \vec{C}$$

Example 2.

$$\int \left(\frac{1}{t+1} \vec{i} + t \cos(t^2 - 1) \vec{j} \right) dt = ?$$

Integration properties

$$\frac{d}{dt} \int \vec{r}(t) dt = \vec{r}(t) \quad \text{and} \quad \int \vec{r}'(t) dt = \vec{r}(t) + \vec{C}$$

Fundamental Theorem of Calculus

$$\int_a^b \vec{r}(t) dt = \vec{R}(t) \Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

Example 3.

$$\int_0^1 \left(\frac{1}{t+1} \vec{i} + t \cos(t^2 - 1) \vec{j} \right) dt = ?$$

Example 4. Find $\vec{r}(t)$ given that

$$\vec{r}'(t) = \left(\frac{1}{t+1}, t \cos(t^2 - 1) \right), \quad \vec{r}(1) = (1, 3)$$

8 Arc Length Parametrisation

We say that $\vec{r}(t)$ is **smoothly parameterised** or that $\vec{r}(t)$ is a **smooth function** of t if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ for any allowed value of t .

Geometrically it implies that the tangent vector $\vec{r}'(t)$ varies continuously along the curve. For this reason a smoothly parameterised function is said to have a **continuously turning tangent vector**.

Example 1.

$$\vec{r}(t) = a \cos t \vec{i} + a \sin t \vec{j} + vt \vec{k}, \quad \vec{r}'(t) = ?$$

Example 2.

$$\vec{r}(t) = t^2 \vec{i} + t^3 \vec{j}, \quad \vec{r}'(t) = ?$$

Change of Parameter

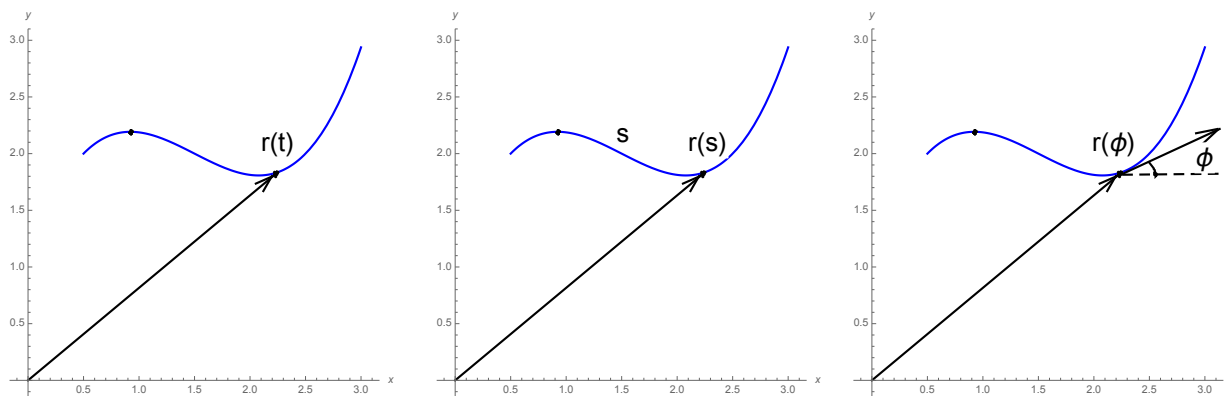


Figure 11: Parameters: time; distance travelled; angle ϕ

A change of parameter in a v-v function $\vec{r}(t)$ is a substitution $t = g(\tau)$ that produces a new v-v function $\vec{r}(g(\tau))$ having the same graph as $\vec{r}(t)$.

Example. Find a change of parameter $t = g(\tau)$ for the circle $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j}$, $0 \leq t \leq 2\pi$ such that the circle is traced (a) counterclockwise; (b) clockwise; as τ increases over the interval $[0, 1]$.

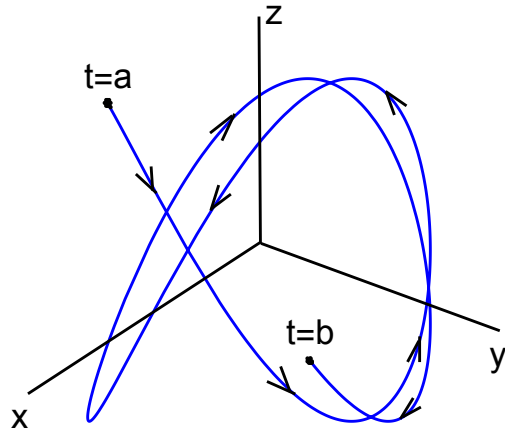
Chain rule:

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt} \frac{dt}{d\tau}, \quad t = g(\tau).$$

A change of parameter $t = g(\tau)$ in which $\vec{r}(g(\tau))$ is smooth if $\vec{r}(t)$ is smooth is called a smooth change of parameter.

It requires $dt/d\tau \neq 0 \forall \tau$ and $dt/d\tau$ continuous. If $dt/d\tau > 0$ it is positive change of parameter, if $dt/d\tau < 0$ it is negative.

Arc Length as a Parameter



The arc length L of a parametric curve

$$x = x(t), y = y(t), z = z(t), a \leq t \leq b:$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

In a vector form $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \implies \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

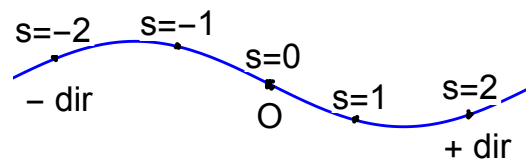
$$\implies L = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b |\vec{r}'(t)| dt.$$

Example.

$$x = a \cos t \quad y = a \sin t \quad z = vt \quad 0 \leq t \leq \pi \quad L = ?$$

The length of arc measured along the curve from some fixed reference point can serve as a parameter.

1. Select an arbitrary point on the curve \mathcal{C} to serve as a **reference point** O .

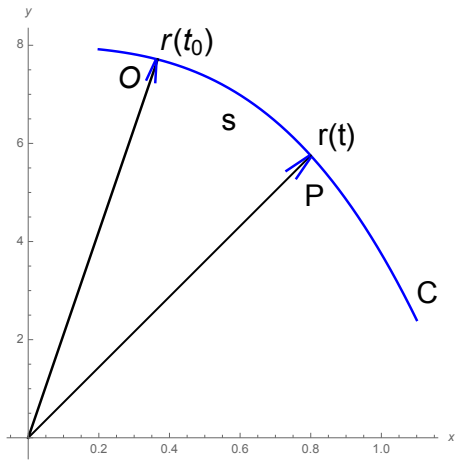


2. Choose one direction along \mathcal{C} to be **positive**, and the other to be **negative**.
3. If P is a point on \mathcal{C} , let s be the “signed” arc length along \mathcal{C} from O to P , where s is positive/negative if P is in +/- parts of \mathcal{C} .
4. $x = x(s)$, $y = y(s)$, $z = z(s)$ is called an arc length parametrisation of the curve. It depends on the reference point and direction.

Example.

$$x = a \cos t, \quad y = a \sin t, \quad 0 \leq t \leq 2\pi$$

Finding arc length parameterisation



$$s = \int_{t_0}^t \left| \frac{d\vec{r}}{du} \right| du$$

Since $\frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| > 0$ it is a positive change of parameter from t to s .

Example 1.

$$\vec{r} = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$$

Example 2. A line through \vec{r}_0 and parallel to \vec{v}

Properties of arc length parameterisation

1. $\forall t, \left| \frac{ds}{dt} \right| = \left| \frac{d\vec{r}}{dt} \right| > 0$ (it is speed).
2. $\forall s$, the tangent vector has length 1: $\left| \frac{d\vec{r}}{ds} \right| = \left| \frac{ds}{ds} \right| = 1$
3. If $\left| \frac{d\vec{r}}{dt} \right| = 1 \forall t$, then $\forall t_0, s = t - t_0$ is the arc length parameter that has the reference point at $t = t_0$.