1 Maxima and minima of functions of two variables

**Definition.** A function $f$ of two variables is said to have a **relative maximum** (minimum) at a point $(a, b)$ if there is a disc centred at $(a, b)$ such that
$$f(a, b) \geq f(x, y) \ (f(a, b) \leq f(x, y))$$
for all points $(x, y)$ that lie inside the disc.

A function $f$ is said to have an **absolute maximum** (minimum) at $(a, b)$ if
$$f(a, b) \geq f(x, y) \ (f(a, b) \leq f(x, y))$$
for all points $(x, y)$ that lie inside in the domain of $f$.

If $f$ has a relative (absolute) maximum or minimum at $(a, b)$ then we say that $f$ has a **relative** (absolute) extremum at $(a, b)$.

relative $\leftrightarrow$ local
The extreme-value theorem. If $f(x, y)$ is continuous on a closed and bounded set $R$, then $f$ has both absolute maximum and an absolute minimum on $R$.

Finding relative extrema

Theorem. If $f$ has a relative extremum at $(a, b)$, and if the first-order derivatives of $f$ exist at this point, then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

Definition. A point $(a, b)$ in the domain of $f(x, y)$ is called a critical point of $f$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one or both partial derivatives do not exist at $(a, b)$.

Example. $f(x, y) = y^2 - x^2$ is a hyperbolic paraboloid.

$f_x = -2x, f_y = 2y \Rightarrow (0, 0)$ is critical but it is not a relative extremum.

It is a saddle point.
We say that a surface $z = f(x, y)$ has a **saddle point** at $(a, b)$ if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at $(a, b)$, and the trace in the other has a relative minimum at $(a, b)$.

**Example.**

- $f(x, y) = x^2 + y^2$
  - $f_x(0, 0) = f_y(0, 0) = 0$
  - Relative and absolute min at $(0, 0)$

- $f(x, y) = 1 - x^2 - y^2$
  - $f_x(0, 0) = f_y(0, 0) = 0$
  - Relative and absolute max at $(0, 0)$

- $f(x, y) = \sqrt{x^2 + y^2}$
  - $f_x(0, 0)$ and $f_y(0, 0)$ do not exist
  - Relative and absolute min at $(0, 0)$

How to determine whether a critical point is a max or min?
The second partials test

**Theorem.** Let $f(x, y)$ have continuous second-order partial
derivatives in some disc centred at a critical point $(a, b)$, and let

\[ D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 \]

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a relative minimum at $(a, b)$.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a relative maximum at $(a, b)$.
3. If $D < 0$, then $f$ has a saddle point at $(a, b)$.
4. If $D = 0$, then no conclusion can be drawn.

**Example.**

\[ f(x, y) = x^4 - x^2y + y^2 - 3y + 4 \]

**How to find the absolute extrema of a continuous function of two
variables on a closed and bounded set $R$?**

1. Find the critical points of $f$ that lie in the interior of $R$.
2. Find all the boundary points at which the absolute extrema can occur.
3. Evaluate $f(x, y)$ at the found points. The largest of these values is the absolute maximum, and the smallest the absolute minimum.

**Example.**

\[ f(x, y) = 3x + 6y - 3xy - 7, \quad R \text{ is the triangle } (0, 0), (0, 3), (5, 0) \]
Lagrange multipliers

Extremum problems with constraints:
Find max or min of the function $f(x_1, \ldots, x_n)$ subject to constraints $g_\alpha(x_1, \ldots, x_n), \alpha = 1, \ldots, m$

Consider $f(x, y)$ and $g(x, y) = 0$.
The graph of $g(x, y) = 0$ is a curve.
Consider level curves of $f$: $f(x, y) = k$.
At $(a, b)$ the curves just touch, and thus have a common tangent line at $(a, b)$. Since $\nabla f(a, b)$ is normal to the level curve at $(a, b)$, and $\nabla g(a, b)$ is normal to the constraint curve at $(a, b)$, we get
$$\nabla f(a, b) \parallel \nabla g(a, b)$$

for some scalar $\lambda$ called the Lagrange multiplier.

**Proof.** Parametrise $g(x, y) = 0$.
Then, $f(x, y) = f(x(t), y(t))$ is a function of $t$ and its local extrema are at
$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y'$$
$$= \nabla f \cdot (x' \hat{i} + y' \hat{j}) = \nabla f \cdot \hat{T}$$
Thus, both $\nabla f$ and $\nabla g$ are $\perp$ to $\hat{T}$. 
In general, we introduce a Lagrange multiplier $\lambda_\alpha$ for each of the constraint $g_\alpha$, and the equations are

$$\vec{\nabla} f = \sum_{\alpha=1}^{m} \lambda_\alpha \vec{\nabla} g_\alpha .$$

**Example.** Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$. 