1 Parametric surfaces

A curve: \( x = x(t), y = y(t), z = z(t), t_0 \leq t \leq t_1 \)

A surface: \( x = x(u, v), y = y(u, v), z = z(u, v), (u, v) \in \mathbb{R} \)
**Example.** $z = 4 - x^2 - y^2$

Rectangular coordinates: $x = u$, $y = v$, $z = 4 - u^2 - v^2$

Polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$, $z = 4 - r^2$
Polar coordinates are useful for **surfaces of revolution** generated by revolving a curve $z = f(x), \ x \geq 0$ in the $xz$-plane, or, equivalently, a curve $z = f(y), \ y \geq 0$ in the $yz$-plane about the $z$-axis.

These surfaces are graphs of functions $z = f(\sqrt{x^2 + y^2})$, and can be parametrised as $x = r \cos \theta, \ y = r \sin \theta, \ z = f(r)$.

**Example.** A right cone of height $h$ and base radius $a$ oriented along the $z$-axis, with vertex pointing up, and with the base located at $z = 0$.

\[ x = r \cos \theta, \ y = r \sin \theta, \ z = h\left(1 - \frac{r}{a}\right), \quad 0 \leq r \leq a, \ 0 \leq \theta \leq 2\pi \]
Cylindrical coordinates

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z \]
\[ r \geq 0, \quad 0 \leq \theta \leq 2\pi \]

Plane: \( z = \text{const} \)
Circular cylinder: \( r = \text{const} \)
Half-plane: \( \theta = \text{const} \)

Cylindrical coordinates are useful for **surfaces of revolution** generated by revolving a curve \( x = f(z) \) in the \( xz \)-plane or, equivalently, a curve \( y = f(z) \) in the \( yz \)-plane about the \( z \)-axis.

These surfaces are parametrised as

\[ x = f(\zeta) \cos \theta, \quad y = f(\zeta) \sin \theta, \quad z = \zeta \]

**Example.** A right cone of height \( h \) and base radius \( a \) oriented along the \( z \)-axis, with vertex pointing up, and with the base located at \( z = 0 \).

\[ x = (1 - \frac{\zeta}{h})a \cos \theta, \quad y = (1 - \frac{\zeta}{h})a \sin \theta, \quad z = \zeta, \quad 0 \leq \zeta \leq h, \quad 0 \leq \theta \leq 2\pi \]
Spherical coordinates

\[ x = \rho \sin \phi \cos \theta, \]
\[ y = \rho \sin \phi \sin \theta, \]
\[ z = \rho \cos \phi, \]
\[ \rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi. \]

Sphere: \( r = \text{const} \)
Half-plane: \( \theta = \text{const} \)
Cone: \( \phi = \text{const} \)

Example. \( x^2 + y^2 + z^2 = 9 \Rightarrow \rho = 3 \)
\[ x = 3 \sin \phi \cos \theta, \quad y = 3 \sin \phi \sin \theta, \quad z = 3 \cos \phi, \]
\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi. \]

The constant \( \phi \)-curves are the lines of \textit{latitude}. The constant \( \theta \)-curves are the lines of \textit{longitude}.  

Example. \( z = \sqrt{x^2+y^2} \Rightarrow \phi = \frac{\pi}{3} \)
\[ x = \frac{\sqrt{3}}{2} \rho \cos \theta, \quad y = \frac{\sqrt{3}}{2} \rho \sin \theta, \quad z = \frac{1}{2} \rho, \]
\[ \rho \geq 0, \quad 0 \leq \theta \leq 2\pi. \]

The constant \( \rho \)-curves are circles.
The constant \( \theta \)-curves are half-lines.
Sphere is an example of a **surface of revolution** generated by revolving a parametric curve

\[ x = f(t), \ z = g(t) \] or, equivalently, a parametric curve \[ y = f(t), \ z = g(t) \] about the \( z \)-axis.

Such a surface is parametrised as

\[ x = f(t) \cos \theta, \ y = f(t) \sin \theta, \ z = g(t). \]

**Example.** Torus.

Similarly, we can get surfaces of revolution by revolving a parametric curve

\[ y = f(u), \ x = g(u) \] or, equivalently, a parametric curve \[ z = f(u), \ x = g(u) \] about the \( x \)-axis.

They are parametrised as

\[
\begin{align*}
x & = g(u), \\
y & = f(u) \cos v, \\
z & = f(u) \sin v
\end{align*}
\]

Finally, we can get surfaces of revolution by revolving a parametric curve \( z = f(u), \ y = g(u) \) or, equivalently, a parametric curve \( x = f(u), \ y = g(u) \) about the \( y \)-axis.

They are parametrised as

\[
\begin{align*}
x & = f(u) \sin v, \\
y & = g(u), \\
z & = f(u) \cos v
\end{align*}
\]
Vector-valued functions of two variables

The vector form of parametric eqs for a surface

\[ \vec{r} = \vec{r}(u,v) = x(u,v) \hat{i} + y(u,v) \hat{j} + z(u,v) \hat{k} \]

\( \vec{r}'(u,v) \) is a vector-valued functions of two variables.

Partial derivatives

\[ \vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} + \frac{\partial z}{\partial u} \hat{k} \]

\[ \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} + \frac{\partial z}{\partial v} \hat{k} \]

Tangent planes to parametric surfaces

Let \( \sigma \) be a parametric surface in 3-space.

**Definition.** A plane is said to be tangent to \( \sigma \) at \( P_0 \) provided a line through \( P_0 \) lies in the plane if and only if it is a tangent line at \( P_0 \) to a curve on \( \sigma \).

\( \sigma : \vec{r}(u,v), \ P_0(a,b,c) \in \sigma \)

\( a = x(u_0,v_0), \ b = y(u_0,v_0), \ c = z(u_0,v_0) \)

If \( \frac{\partial \vec{r}}{\partial u} \neq 0 \) then it is tangent to the constant \( v \)-curve.

If \( \frac{\partial \vec{r}}{\partial v} \neq 0 \) then it is tangent to the constant \( u \)-curve.

Thus, if \( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0 \) at \( (u_0,v_0) \) then it is orthogonal to both tangent vectors and is normal to the tangent plane and the surface at \( P_0 \).
\[ \vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} \]

is called the **principle normal vector** to the surface \( \vec{r} = \vec{r}(u,v) \) at \((u_0, v_0)\).

Thus, the tangent plane equation is

\[ \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \]

**Example.** Tangent plane to 

\[ x = uv, \; y = u, \; z = v^2 \] at \((2, -1)\)

![Figure 1: Whitney’s umbrella.](image)

*Answer:* \( x + y + z = 1 \)
Surface area

Let \( \mathbf{r} = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \) be a smooth parametric surface on a region \( R \) of the \( uv\)-plane, that is \( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq 0 \) on \( R \), and therefore there is a tangent plane for every \((u, v)\) \( \in R \).

\[ R_k : \quad \Delta A_k = \Delta u_k \Delta v_k \]

\[ \Delta S_k \approx \text{Area of parallelogram} = \left| \frac{\partial \mathbf{r}}{\partial u} \Delta u_k \times \frac{\partial \mathbf{r}}{\partial v} \Delta v_k \right| \]

\[ = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u_k \Delta v_k = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta A_k \]

Thus,

\[ S \approx \sum_{k=1}^{n} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta A_k \]

In the limit \( n \to \infty \)

\[ S = \iint_{R} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA \]

Examples. Surfaces of revolution: Sphere, cone, torus
Surface area of surfaces of the form $z = f(x, y)$

$x = u, y = v, z = f(u, v), \quad \vec{r} = u\hat{i} + v\hat{j} + z\hat{k}$

\[
\frac{\partial \vec{r}}{\partial u} = \hat{i} + \frac{\partial z}{\partial u} \hat{k}, \quad \frac{\partial \vec{r}}{\partial v} = \hat{j} + \frac{\partial z}{\partial v} \hat{k}
\]

\[
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \hat{k} - \frac{\partial z}{\partial v} \hat{j} - \frac{\partial z}{\partial u} \hat{i}
\]

\[
\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{1 + \left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2}
\]

\[
S = \iint_{R} \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA
\]

Example. $S$ of $z = x^2 + y^2$ below $z = 1$.
Answer: $\frac{\pi}{6}(5\sqrt{5} - 1) \approx 5.330$