Multiple Integrals

1 Double Integrals

Definite integrals appear when one solves

**Area problem.** Find the area $A$ of the region $R$ bounded above by the curve $y = f(x)$, below by the $x$-axis, and on the sides by $x = a$ and $x = b$.

\[
A = \int_{a}^{b} f(x) \, dx = \lim_{\max \Delta x_i \to 0} \sum_{k=1}^{n} f(x_{k}^{*}) \Delta x_k
\]

**Mass problem.** Find the mass $M$ of a rod of length $L$ whose linear density (the mass per unit length) is a function $\delta(x)$, where $x$ is the distance from a point of the rod to one of the rod’s ends.

\[
M = \int_{0}^{L} \delta(x) \, dx
\]
Double integrals appear when one solves

**Volume problem.** Find the volume $V$ of the solid $G$ enclosed between the surface $z = f(x, y)$ and a region $R$ in the $xy$-plane where $f(x, y)$ is continuous and nonnegative on $R$.

**Mass problem.** Find the mass $M$ of a lamina (a region $R$ in the $xy$-plane) whose density (the mass per unit area) is a continuous nonnegative function $\delta(x, y)$ defined as

$$
\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A}
$$

where $\Delta M$ is the mass of the small rectangle of area $\Delta A$ which contains $(x, y)$.
Let us consider the volume problem.

1. Divide the rectangle enclosing $R$ into subrectangles, and exclude all those rectangles that contain points outside of $R$. Let $n$ be the number of all the rectangles inside $R$, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the $k$-th subrectangle.

2. Choose any point $(x_k^*, y_k^*)$ in the $k$-th subrectangle. The volume of a rectangular parallelepiped with base area $\Delta A_k$ and height $f(x_k^*, y_k^*)$ is $\Delta V_k = f(x_k^*, y_k^*) \Delta A_k$. Thus,

   \[ V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k = \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta x_k \Delta y_k \]

   This sum is called the Riemann sum.

3. Take the sides of all the subrectangles to 0, and therefore the number of them to infinity, and get

   \[ V = \lim_{\max \Delta x_i, \Delta y_i \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k = \iint_R f(x, y) \, dA \]

   The last term is the notation for the limit of the Riemann sum, and it is called the double integral of $f(x, y)$ over $R$. 

In what follows we identify

\[
\lim_{\Delta x_i, \Delta y_i \to 0} \max \sum_{k=1}^{n} \cdots \equiv \lim_{n \to \infty} \sum_{k=1}^{n} \cdots
\]

If \( f \) is continuous but not nonnegative on \( R \) then the limit represents a difference of volumes – above and below the \( xy \)-plane. It is called the net signed volume between \( R \) and the surface \( z = f(x, y) \), and it is given by the limit of the corresponding Riemann sum that is the double integral of \( f(x, y) \) over \( R \)

\[
\iint_{R} f(x, y) \, dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta x_k \Delta y_k
\]

Similarly, the mass \( M \) of a lamina with density \( \delta(x, y) \) is

\[
M = \iint_{R} \delta(x, y) \, dA
\]
Properties of double integrals

1. If $f, g$ are continuous on $R$, and $c, d$ are constants, then

$$\iint_R \left( c \, f(x, y) + d \, g(x, y) \right) \, dA = c \iint_R f(x, y) \, dA + d \iint_R g(x, y) \, dA$$

2. If $R$ is divided into two regions $R_1$ and $R_2$, then

$$\iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA$$

The volume of the entire solid is the sum of the volumes of the solids above $R_1$ and $R_2$. 
Double integrals over rectangular regions

The symbols
\[
\int_a^b f(x, y) \, dx \quad \text{and} \quad \int_c^d f(x, y) \, dy
\]
where in the first integral \( y \) is fixed while in the second integral \( x \) is fixed, denote \textbf{partial definite integrals}.

Examples.

\[
(i) \int_1^2 \sin(2x - 3y) \, dx, \quad (ii) \int_0^1 \sin(2x - 3y) \, dy.
\]

We can then integrate the resulting functions of \( y \) and \( x \) with respect to \( y \) and \( x \), respectively. This two-stage integration process is called \textbf{iterated or repeated integration}.

Notation
\[
\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy
\]
\[
\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx
\]

These integrals are called \textbf{iterated integrals}.

Example.

\[
(i) \int_0^1 \int_1^2 \sin(2x - 3y) \, dx \, dy, \quad (ii) \int_1^2 \int_0^1 \sin(2x - 3y) \, dy \, dx
\]
Theorem. Let $R$ be the rectangle $a \leq x \leq b$, $c \leq y \leq d$. If $f(x, y)$ is continuous on $R$ then

$$\iint_{R} f(x, y) \, dA = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx$$

Example. Use a double integral to find $V$ under the surface

$$z = 3\pi e^{x} \sin y + e^{-x}$$

and over the rectangle

$$R = \{(x, y) : 0 \leq x \leq \ln 3, 0 \leq y \leq \pi\}$$

$$V = \frac{38}{3} \pi \approx 39.7935 > 0$$
Double integrals over nonrectangular regions

\[ \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left[ \int_c^{g_2(x)} f(x, y) \, dy \right] \, dx \]

Replace \( c \to g_1(x) \), \( d \to g_2(x) \). Then

\[ \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right] \, dx \]

Figure 1: The rectangle becomes a type I region: \( g_2(x) \geq g_1(x) \).

**Theorem.** If \( R \) is a type I region on which \( f(x, y) \) is continuous, then

\[ \iiint_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \]

**Example.** Find the volume \( V \) of the solid enclosed by the surfaces \( z = 0 \), \( y^2 = x \), and \( x + z = 1 \).

\[ V = \frac{8}{15} \]
Similarly, in
\[
\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = \int_{c}^{d} \left[ \int_{a}^{b} f(x, y) \, dx \right] \, dy
\]
we replace \(a \rightarrow h_1(y), b \rightarrow h_2(y)\). Then
\[
\int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_{c}^{d} \left[ \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \right] \, dy
\]

![Diagram showing a type II region](image)

**Figure 2:** The rectangle becomes a type II region: \(h_2(y) \geq h_1(y)\).

**Theorem.** If \(R\) is a type II region on which \(f(x, y)\) is continuous, then
\[
\iiint_{R} f(x, y) \, dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy
\]

**Example.** Find the volume \(V\) of the solid enclosed by the surfaces \(z = 0, \ y^2 = x, \) and \(x + z = 1\).
Some regions belong to both type I and II, e.g. the region from the previous example, a disc and a triangle.

**Reversing the order of integration. Example.**

\[
\int_{0}^{4} \int_{\sqrt{y}}^{2} e^{x^3} \, dx \, dy = \frac{1}{3} (e^8 - 1)
\]

**Area calculated as a double integral**

The solid enclosed between the plane \( z = 1 \), and a region \( R \) is a right cylinder with base \( R \), height \( h = 1 \), and with cross-sectional area \( A \) equal to the area of \( R \). Its volume is

\[
V = A \, h = A = \iint_{R} 1 \, dA
\]

Thus

\[
\text{Area of } R = \iint_{R} 1 \, dA = \iiint_{R} dA
\]

**Example.** Find \( A \) of \( R \) enclosed between \( y = x^m \) and \( y = x^n \).
3 Double integrals in polar coordinates

Polar coordinates

Figure 3: $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$.

We identify $(r, \theta + 2\pi n) \sim (r, \theta)$, $n = \pm 1, \pm 2, \ldots$
and $(-r, \theta) \sim (r, \theta + \pi)$
because they give the same $x, y$ coordinates.

The graph of $r = a$ is the circle of radius $a$ centred at $O$.
The graph of $\theta = \alpha$, $r \geq 0$ is the ray making an angle of $\alpha$ with the polar axis.

A polar rectangle is a region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two circles $r = a$, $r = b$.

Its area is
\[
A = \frac{1}{2} b^2 (\beta - \alpha) - \frac{1}{2} a^2 (\beta - \alpha)
= \frac{1}{2} (b + a) (b - a) (\beta - \alpha)
= \bar{r} \, \Delta r \, \Delta \theta, \text{ where } \bar{r} = \frac{1}{2} (b + a),
\Delta r = b - a \text{ is the radial thickness, } \Delta \theta = \beta - \alpha \text{ is the central angle.}
A simple polar region
is a region enclosed between
two rays, \( \theta = \alpha, \theta = \beta \), and
two continuous polar curves
\( r = r_1(\theta), r = r_2(\theta) \) which satisfy
(i) \( \alpha \leq \beta \),
(ii) \( \beta - \alpha \leq 2\pi \),
(iii) \( 0 \leq r_1(\theta) \leq r_2(\theta) \).

Examples.

Figure 4: \( r_1(\theta) = 0 \) and \( \beta - \alpha < 2\pi \)

\( r_1(\theta) = 0 \) and \( \beta - \alpha = 2\pi \) \hspace{1cm} \( r_1(\theta) \neq 0 \) and \( \beta - \alpha = 2\pi \)
Let us consider the volume problem in polar coordinates.

1. Divide the rectangle enclosing $R$ into **polar** subrectangles, and exclude all those rectangles that contain points outside of $R$. Let $n$ be the number of all the rectangles inside $R$, and let

$$\Delta A_k = \bar{r}_k \Delta r_k \Delta \theta_k$$

be the area of the $k$-th polar subrectangle.

2. Choose any point $(r_k^*, \theta_k^*)$ in the $k$-th subrectangle. The volume of a right cylinder with base area $\Delta A_k$ and height $f(r_k^*, \theta_k^*)$ is

$$\Delta V_k = f(r_k^*, \theta_k^*) \Delta A_k.$$ 

Thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \bar{r}_k \Delta r_k \Delta \theta_k$$

This sum is called the **polar Riemann sum**.

3. Take the sides of all the subrectangles to 0, and therefore the number of them to infinity, and get

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k = \iint_R f(r, \theta) \, dA$$

The last term is the notation for the limit of the Riemann sum, and it is called the **polar double integral** of $f(r, \theta)$ over $R$. 
The limit of the polar Riemann sum is the same for any choice of points \((r_k^*, \theta_k^*)\). Let us choose \((r_k^*, \theta_k^*)\) to be the centre of the \(k\)-th polar rectangle, that is \(r_k^* = \bar{r}_k, \theta_k^* = \bar{\theta}_k = \frac{1}{2}(\theta_k - 1 + \theta_k)\). Then, the polar double integral is given by

\[
\iint_R f(r, \theta) \, dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{r}_k, \bar{\theta}_k) \bar{r}_k \Delta r_k \Delta \theta_k
\]

This formula is similar to the one for the double integral in rectangular coordinates, and it is valid for any region \(R\).

**Theorem.** If \(R\) is a simple polar region enclosed between two rays, \(\theta = \alpha, \theta = \beta\), and two continuous polar curves \(r = r_1(\theta), r = r_2(\theta)\), and if \(f(r, \theta)\) is continuous on \(R\), then

\[
\iint_R f(r, \theta) \, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) \, r \, dr \, d\theta
\]

**Example.** Find the volume of the solid below \(z = 1 - x^2 - y^2\), inside of \(x^2 + y^2 - x = 0\), and above \(z = 0\). *Answer: \(V = 5\pi/32\)

Area of \(R = \iint_R dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} r \, dr \, d\theta = \frac{1}{2} \int_{\alpha}^{\beta} (r_2(\theta)^2 - r_1(\theta)^2) \, d\theta
\]

**Example.** Find \(A\) of \(R\) in the first quadrant that is outside \(r = 2\) and inside the cardioid \(r = 2(1 + \cos \theta)\). *Answer: \(A = 4 + \pi/2\)

**Example.** Find \(A\) enclosed by the three-petaled rose \(r = \cos 3\theta\). *Answer: \(A = \pi/4\)
Converting double integrals from rectangular to polar coordinates

\[ \iint_R f(x, y) \, dA = \iint f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \]

It is especially useful if \( f(x, y) = g(x^2 + y^2) = g(r) \)
or \( f(x, y) = g(y/x) = g(\tan \theta) \)

Example.

\[ \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx = \frac{\pi}{5} \]

Example.

\[ \int_{0}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx + \int_{1}^{2} \int_{0}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx = \frac{3}{2} \]

Example.

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]
4 Parametric surfaces

A curve: $x = x(t), \ y = y(t), \ z = z(t), \ t_0 \leq t \leq t_1$

A surface: $x = x(u, v), \ y = y(u, v), \ z = z(u, v), \ (u, v) \in \mathbb{R}$
**Example.** \( z = 4 - x^2 - y^2 \)

Rectangular coordinates: \( x = u, \ y = v, \ z = 4 - u^2 - v^2 \)

Polar coordinates: \( x = r \cos \theta, \ y = r \sin \theta, \ z = 4 - r^2 \)
Polar coordinates are useful for **surfaces of revolution** generated by revolving a curve \( z = f(x), \ x \geq 0 \) in the \( xz \)-plane, or, equivalently, a curve \( z = f(y), \ y \geq 0 \) in the \( yz \)-plane about the \( z \)-axis.

These surfaces are graphs of functions \( z = f(\sqrt{x^2 + y^2}) \), and can be parametrised as

\[
x = r \cos \theta, \ y = r \sin \theta, \ z = f(r)
\]

**Example.** A right cone of height \( h \) and base radius \( a \) oriented along the \( z \)-axis, with vertex pointing up, and with the base located at \( z = 0 \).

\[
x = r \cos \theta, \ y = r \sin \theta, \ z = h(1 - \frac{r}{a}), \ 0 \leq r \leq a, \ 0 \leq \theta \leq 2\pi
\]
Cylindrical coordinates

\[ x = r \cos \theta, \ y = r \sin \theta, \ z \]
\[ r \geq 0, \ 0 \leq \theta \leq 2\pi \]

Plane: \( z = \text{const} \)
Circular cylinder: \( r = \text{const} \)
Half-plane: \( \theta = \text{const} \)

Cylindrical coordinates are useful for **surfaces of revolution** generated by revolving a curve \( x = f(z) \) in the \( xz \)-plane or, equivalently, a curve \( y = f(z) \) in the \( yz \)-plane about the \( z \)-axis.

These surfaces are parametrised as

\[ x = f(\zeta) \cos \theta, \ y = f(\zeta) \sin \theta, \ z = \zeta \]

**Example.** A right cone of height \( h \) and base radius \( a \) oriented along the \( z \)-axis, with vertex pointing up, and with the base located at \( z = 0 \).

\[ x = (1 - \frac{\zeta}{h})a \cos \theta, \ y = (1 - \frac{\zeta}{h})a \sin \theta, \ z = \zeta, \ \ 0 \leq \zeta \leq h, \ 0 \leq \theta \leq 2\pi \]
Spherical coordinates

\[ x = \rho \sin \phi \cos \theta , \]
\[ y = \rho \sin \phi \sin \theta , \]
\[ z = \rho \cos \phi , \]
\[ \rho \geq 0 , \ 0 \leq \theta \leq 2\pi , \]
\[ 0 \leq \phi \leq \pi . \]

Sphere: \( r = \text{const} \)

Half-plane: \( \theta = \text{const} \)

Cone: \( \phi = \text{const} \)

Example. \( x^2 + y^2 + z^2 = 9 \Rightarrow \rho = 3 \)
\[ x = 3 \sin \phi \cos \theta , \ y = 3 \sin \phi \sin \theta , \ z = 3 \cos \phi , \]
\[ 0 \leq \theta \leq 2\pi , \ 0 \leq \phi \leq \pi . \]

The constant \( \phi \)-curves are the lines of latitude.

The constant \( \theta \)-curves are the lines of longitude.

Example. \( z = \sqrt{\frac{x^2+y^2}{3}} \Rightarrow \phi = \frac{\pi}{3} \)
\[ x = \frac{\sqrt{3}}{2} \rho \cos \theta , \ y = \frac{\sqrt{3}}{2} \rho \sin \theta , \ z = \frac{1}{2} \rho , \]
\[ \rho \geq 0 , \ 0 \leq \theta \leq 2\pi . \]

The constant \( \rho \)-curves are circles.

The constant \( \theta \)-curves are half-lines.
Sphere is an example of a **surface of revolution** generated by revolving a parametric curve
\( x = f(t), \ z = g(t) \) or, equivalently, a parametric curve \( y = f(t), \ z = g(t) \) about the \( z \)-axis.

Such a surface is parametrised as
\[
\begin{align*}
x &= f(t) \cos \theta, \\
y &= f(t) \sin \theta, \\
z &= g(t).
\end{align*}
\]

**Example.** Torus.

Similarly, we can get surfaces of revolution by revolving a parametric curve
\( y = f(u), \ x = g(u) \) or, equivalently, a parametric curve \( z = f(u), \ x = g(u) \)
about the \( x \)-axis.

They are parametrised as
\[
\begin{align*}
x &= g(u), \\
y &= f(u) \cos v, \\
z &= f(u) \sin v.
\end{align*}
\]

Finally, we can get surfaces of revolution by revolving a parametric curve \( z = f(u), \ y = g(u) \) or, equivalently, a parametric curve \( x = f(u), \ y = g(u) \) about the \( y \)-axis.

They are parametrised as
\[
\begin{align*}
x &= f(u) \sin v, \\
y &= g(u), \\
z &= f(u) \cos v.
\end{align*}
\]
Vector-valued functions of two variables

The vector form of parametric eqs for a surface

\[ \vec{r} = \vec{r}(u, v) = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k} \]

\( \vec{r}(u, v) \) is a vector-valued functions of two variables.

Partial derivatives

\[ \vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} + \frac{\partial z}{\partial u} \hat{k} \]

\[ \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} + \frac{\partial z}{\partial v} \hat{k} \]

Tangent planes to parametric surfaces

Let \( \sigma \) be a parametric surface in 3-space.

Definition. A plane is said to be tangent to \( \sigma \) at \( P_0 \) provided a line through \( P_0 \) lies in the plane if and only if it is a tangent line at \( P_0 \) to a curve on \( \sigma \).

\[ \sigma : \vec{r}(u, v) , \quad P_0(a, b, c) \in \sigma \]

\[ a = x(u_0, v_0) , \quad b = y(u_0, v_0) , \quad c = z(u_0, v_0) \]

If \( \frac{\partial \vec{r}}{\partial u} \neq 0 \) then it is tangent to the constant \( v \)-curve.

If \( \frac{\partial \vec{r}}{\partial v} \neq 0 \) then it is tangent to the constant \( u \)-curve.

Thus, if \( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0 \) at \((u_0, v_0)\) then it is orthogonal to both tangent vectors and is normal to the tangent plane and the surface at \( P_0 \).
\[ \vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \]

is called the \textbf{principle normal vector} to the surface \( \vec{r} = \vec{r}(u, v) \) at \((u_0, v_0)\).
Thus, the tangent plane equation is

\[ \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \]

\textbf{Example.} Tangent plane to

\[ x = uv, \ y = u, \ z = v^2 \] at \((2, -1)\)

\[ \text{Answer: } x + y + z = 1 \]
**Surface area**

Let \( \vec{r} = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k} \) be a smooth parametric surface on a region \( R \) of the \( uv\)-plane, that is \( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0 \) on \( R \), and therefore there is a tangent plane for every \((u, v) \in R\).

\[ R_k : \Delta A_k = \Delta u_k \Delta v_k \]

\[ \Delta S_k \approx \text{Area of parallelogram} = \left| \frac{\partial \vec{r}}{\partial u} \Delta u_k \times \frac{\partial \vec{r}}{\partial v} \Delta v_k \right| \]

\[ = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u_k \Delta v_k = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta A_k \]

Thus,

\[ S \approx \sum_{k=1}^{n} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta A_k \]

In the limit \( n \to \infty \)

\[ S = \iint_R \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA \]

**Examples.** Surfaces of revolution: Sphere, cone, torus
Surface area of surfaces of the form $z = f(x, y)$

$$x = u, \ y = v, \ z = f(u, v), \ \vec{r} = u\hat{i} + v\hat{j} + z\hat{k}$$

$$\frac{\partial \vec{r}}{\partial u} = \hat{i} + \frac{\partial z}{\partial u} \hat{k}, \quad \frac{\partial \vec{r}}{\partial v} = \hat{j} + \frac{\partial z}{\partial v} \hat{k}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \hat{k} - \frac{\partial z}{\partial v} \hat{j} - \frac{\partial z}{\partial u} \hat{i}$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{1 + \left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2}$$

$$S = \iint_R \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \ dA$$

**Example.** $S$ of $z = x^2 + y^2$ below $z = 1$.

*Answer: $\frac{\pi}{6} (5\sqrt{5} - 1) \approx 5.330$*
5 Triple Integrals

**Mass problem.** Find the mass $M$ of a solid $G$ whose density (the mass per unit volume) is a continuous nonnegative function $\delta(x, y, z)$.

1. Divide the box enclosing $G$ into subboxes, and exclude all those subboxes that contain points outside of $G$. Let $n$ be the number of all the subboxes inside $G$, and let $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$ be the volume of the $k$-th subbox.

2. Choose any point $(x^*_k, y^*_k, z^*_k)$ in the $k$-th subbox. The mass of the $k$-th subbox is $\Delta M_k \approx \delta(x^*_k, y^*_k, z^*_k) \Delta V_k$. Thus,

$$M \approx \sum_{k=1}^{n} \Delta M_k = \sum_{k=1}^{n} \delta(x^*_k, y^*_k, z^*_k) \Delta V_k = \sum_{k=1}^{n} \delta(x^*_k, y^*_k, z^*_k) \Delta x_k \Delta y_k \Delta z_k$$

This sum is called the **Riemann sum**.

3. Take the sides of all the subboxes to $0$, and therefore the number of them to infinity, and get

$$M = \lim_{n \to \infty} \sum_{k=1}^{n} \delta(x^*_k, y^*_k, z^*_k) \Delta V_k = \iiint_{G} \delta(x, y, z) \, dV$$

The last term is the notation for the limit of the Riemann sum, and it is called the **triple integral** of $\delta(x, y, z)$ over $G$. 
Properties of triple integrals

1. If $f, g$ are continuous on $G$, and $c, d$ are constants, then

$$\iiint_G \left( cf(x, y, z) + dg(x, y, z) \right) \, dV = c \iiint_G \! f(x, y, z) \, dV + d \iiint_G \! g(x, y, z) \, dV$$

2. If $G$ is divided into two solids $G_1$ and $G_2$, then

$$\iiint_G \! f(x, y, z) \, dV = \iiint_{G_1} \! f(x, y, z) \, dV + \iiint_{G_2} \! f(x, y, z) \, dV$$

The mass of the entire solid is the sum of the masses of the solids $G_1$ and $G_2$. 
Evaluating triple integrals over rectangular boxes

\[ a \leq x \leq b, \quad c \leq y \leq d, \quad k \leq z \leq l \]

\[ \iiint_G f(x, y, z) \, dV = \int_a^b \int_c^d \int_k^l f(x, y, z) \, dz \, dy \, dx \]

or any permutation, e.g.

\[ \iiint_G f(x, y, z) \, dV = \int_c^d \int_k^l \int_a^b f(x, y, z) \, dx \, dz \, dy \]

**Example.** Find the mass of the box

\[ \frac{1}{3} \leq x \leq \frac{1}{2}, \quad 0 \leq y \leq \pi, \quad 0 \leq z \leq 1 \]

if its density is

\[ \delta(x, y, z) = xz \sin(xy) \]

**Answer:** 
\[ M = \frac{1}{12} + \frac{\sqrt{3}}{4\pi} - \frac{1}{2\pi} \approx 0.0620106 \]
Evaluating triple integrals over simple $xy$-, $xz$-, $yz$-solids

A solid $G$ is called a simple $xy$-solid if it is bounded above by a surface $z = g_2(x, y)$, below by a surface $z = g_1(x, y)$, and its projection on the $xy$ plane is a region $R$.

$$\iiint_G f(x, y, z)\,dV = \iint_R \left[ \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\,dz \right] \,dA$$

**Example.** Find the mass of the solid $G$ defined by the inequalities

$$\frac{\pi}{6} \leq y \leq \frac{\pi}{2}, \quad y \leq x \leq \frac{\pi}{2}, \quad 0 \leq z \leq xy$$

if its density is

$$\delta(x, y, z) = \cos\left(\frac{z}{y}\right)$$

**Answer:** $M = \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \approx 0.442972$
Volume of $G = \iiint_G dV$

**Example.** Find $V$ of $G$ bounded by the surfaces

$$y = x^2, \quad y + z = 4, \quad z = 0$$

*Answer:* $V = \frac{256}{15}$

**Integration in other orders**

A simple $xz$-solid

$$\iiint_G f(x, y, z) \, dV = \iint_R \left[ \int_{g_2(x, z)}^{g_1(x, z)} f(x, y, z) \, dy \right] \, dA$$

A simple $yz$-solid

$$\iiint_G f(x, y, z) \, dV = \iint_R \left[ \int_{g_2(y, z)}^{g_1(y, z)} f(x, y, z) \, dx \right] \, dA$$
6 Centre of gravity and centroid

Mass, centre of gravity and centroid of a lamina

Recall that a lamina is an idealised flat object that is thin enough to be viewed as a 2-d plane region $R$.

The mass $M$ of a lamina with density $\delta(x, y)$ is

$$M = \iint_R \delta(x, y) \, dA$$

The centre of gravity of a lamina is a unique point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force acting at the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{M} \iint_R x \, \delta(x, y) \, dA, \quad \bar{y} = \frac{1}{M} \iint_R y \, \delta(x, y) \, dA$$

**Example.** Find the centre of gravity of a lamina with density $\delta(x, y) = x + 1$ bounded by $x^2 + (y + 1)^2 = 1$

*Answer:* $M = \pi$, $\bar{x} = 1/4$, $\bar{y} = -1$
For a **homogeneous** lamina with \( \delta(x, y) = \text{const} \), the centre of gravity is called the **centroid** of the lamina or the centroid of the region \( R \) because it does not depend on \( \delta(x, y) = \text{const} \).

\[
\bar{x} = \frac{1}{A} \iint_R x \, dA, \quad \bar{y} = \frac{1}{A} \iint_R y \, dA, \quad A = \iint_R dA
\]

**Example.** Find the centroid of a region bounded by

\[(x - 1)^2 + y^2 = 1, \quad \text{and} \quad (x - 2)^2 + y^2 = 4\]

*Answer:* \( A = 3\pi, \quad \bar{x} = 7/3, \quad \bar{y} = 0 \)
Mass, centre of gravity and centroid of a solid

The **centre of gravity** of a solid $G$ with density $\delta(x, y, z)$ is a unique point $(\bar{x}, \bar{y}, \bar{z})$ such that the effect of gravity on the solid is equivalent to that of a single force acting at the point $(\bar{x}, \bar{y}, \bar{z})$

\[
\bar{x} = \frac{1}{M} \iiint_G x \delta(x, y, z) \, dV, \quad \bar{y} = \frac{1}{M} \iiint_G y \delta(x, y, z) \, dV
\]

\[
\bar{z} = \frac{1}{M} \iiint_G z \delta(x, y, z) \, dV, \quad M = \iiint_G \delta(x, y, z) \, dV
\]

**Example.** Find the centre of gravity of a solid $G$ bounded by the surfaces $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above by the surface $z = 5 - x^2 - y^2$, and below by the surface $z = 0$ with the density

\[\delta(x, y, z) = e^{5-x^2-y^2-z}\]

*Answer:* $M = \pi(e^4 - e - 3) \approx 153.561$, $\bar{z} = \frac{2e^4-2e-21}{2(e^4-e-3)} \approx 0.85$

For a **homogeneous** solid with $\delta(x, y) = \text{const}$, the centre of gravity is called the **centroid** of the solid.

\[
\bar{x} = \frac{1}{V} \iiint_G x \, dV, \quad \bar{y} = \frac{1}{V} \iiint_G y \, dV
\]

\[
\bar{z} = \frac{1}{V} \iiint_G z \, dV, \quad V = \iiint_G \, dV
\]

**Example.** Find the centroid of the solid below $z = 10 - x^2 - y^2$, inside of $x^2 + y^2 = 1$, and above $z = 0$

*Answer:* $V = 19\pi/2$, $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 271/57 \approx 4.75439$
7 Triple integrals in cylindrical and spherical coordinates

Cylindrical coordinates

Cylindrical wedge or cylindrical element of volume is interior of intersection of two cylinders: \( r = r_1, r = r_2 \)
two half-planes: \( \theta = \theta_1, \theta = \theta_2 \)
two planes: \( z = z_1, z = z_2 \)

The dimensions: \( \theta_2 - \theta_1, r_2 - r_1, z_2 - z_1 \) are called the central angle, thickness and height of the wedge.

Divide \( G \) by cylindrical wedges

\[
\iiint_G f(r, \theta, z) \, dV = \lim_{n \to \infty} \sum_{n=1}^{\infty} f(r^*_k, \theta^*_k, z^*_k) \Delta V_k
\]

\[
\Delta V_k = \left[ \text{area of base} \right] \cdot \left[ \text{height} \right] = r^*_k \Delta r_k \Delta \theta_k \Delta z_k
\]
**Theorem.**

Let $G$ be a solid whose upper surface is $z = g_2(r, \theta)$ and whose lower surface is $z = g_1(r, \theta)$ in cylindrical coordinates. If projection of $G$ on the $xy$-plane is a simple polar region $R$, and if $f(r, \theta, z)$ is continuous on $G$, then

\[
\iiint_G f(r, \theta, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r, \theta)}^{g_2(r, \theta)} f(r, \theta, z) \, r \, dz \, dr \, d\theta
\]

**Example.** $V$ and centroid of $G$ bounded above by $z = \sqrt{25 - x^2 - y^2}$, below by $z = 0$, and laterally by $x^2 + y^2 = 9$.

*Answer:* $V = \frac{122\pi}{3}, \bar{z} = \frac{1107}{488}$

**Converting triple integrals from rectangular to cylindrical coordinates**

\[
\iiint_G f(x, y, z) \, dV = \iiint g(r, \theta, z) \, r \, dz \, dr \, d\theta
\]

**Example.**

\[
\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz \, dy \, dx = \frac{243}{4\pi}
\]
**Spherical coordinates**

**Spherical wedge** or spherical element of volume is interior of intersection of two spheres: \( \rho = \rho_1, \rho = \rho_2 \)
two half-planes: \( \theta = \theta_1, \theta = \theta_2 \)
nappes of two right circular cones: \( \phi = \phi_1, \phi = \phi_2 \)

The numbers:
\( \theta_2 - \theta_1, \rho_2 - \rho_1, \phi_2 - \phi_1 \)
are the dimensions of the wedge.

Divide \( G \) by spherical wedges

\[
\iiint_G f(r, \theta, \phi) \, dV = \lim_{n \to \infty} \sum_{n=1}^{\infty} f(r^*_k, \theta^*_k, \phi^*_k) \Delta V_k
\]

\[
\Delta V_k = (\rho^*_k)^2 \sin \phi^*_k \Delta \rho_k \Delta \phi_k \Delta \theta_k
\]

\[
\iiint_G f(r, \theta, \phi) \, dV = \iiint f(r, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

appropriate limits
Example. \( V \) and centroid of \( G \) bounded above by 
\[
x^2 + y^2 + z^2 = 16 \text{ and below by } z = \sqrt{x^2 + y^2}.
\]
Answer: \( V = 64(2 - \sqrt{2})\pi/3 > 0, \bar{z} = 3/2(2 - \sqrt{2}) \approx 2.56 \)

Converting triple integrals from rectangular to spherical coordinates

\[
\iiint_G f(x, y, z) \, dV = \iiint f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

Example.

\[
\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx = \frac{64}{9} \pi
\]