A Conversational Approach to the Inverse Function Theorem

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First of all, we need to know what it is that we need to prove. This is the statement of the theorem:

Let $M \supset V \xrightarrow{f} N$ be a c^r function on V which is open in M, a finite dimensional real vector space.
Let $\underline{a} \in V$. Then $\exists$ an open NBD of $\underline{\text{a}}$, call it W, s.t. $W \xrightarrow{f} f(W)$ is a diffeomorphism onto $f(W)$ which is open in V.

Ok, so now we begin the proof. First off we know very little about f so we define a function F, which has the same properties mentioned above, but we can work out how it behaves around the origin.

Proof: Let T be the inverse of $f'(a)$ and put $F(x) = Tf(x+a) - Tf(a)$
Now we have a look at the behavior of F:

$$\begin{aligned}
&F(0) = Tf(0+a) - Tf(a) = 0\\
&F'(x) = Tf'(x+a)\\
&F'(0) = Tf'(0+a) = \mathbb{1}
\end{aligned}$$

This is all well and good, but we need a context in which we can use this. For this we define a closed ball of radius r as follows:

Choose a radius $r > 0$ s.t. the closed ball

$$B = \{x \in M | \, \|x\| \leq r\}$$

Now that we have this, we can make some useful statements (by continuity of F'):

$$\forall x \in B \begin{cases} \|\mathbb{1} - F'(x)\| \leq r \\ detF'(x) \neq 0 \end{cases}$$

Now we use this and the mean value theorem (MVT) to show that F is *injective* on B:

$$\begin{aligned}
&\|x-y\| - \|F(x) - F(y)\| \leq \|(\mathbb{1} - F)x - (\mathbb{1} - F)y\| \leq \frac{1}{2}\|x-y\|\\
&\therefore \|F(x) - F(y)\| \geq \frac{1}{2}\|x-y\|
\end{aligned}$$

So F is injective on B. That's nice, what use is that?
Well, it means that $B \xrightarrow{F} F(B)$ has a continuous inverse which in turn means that $B \xrightarrow{F} F(B)$ is a *homeomorphism.* A homeomorphism is simply a mapping that is bijective, continuous and has a continuous inverse. An umbrella term of sorts.
The next step is to show that $\frac{1}{2}B \subset F(B)$. To do this, we will have to define a suitable function, and use it to define a sequence with a fixed point that somehow involves a point in $\frac{1}{2}B$. Don't worry, it'll be much clearer after we show how it works.

So we need a point in $\frac{1}{2}B$. Let it be $a \in \frac{1}{2}B$. Now define $g(x) = x - F(x) + a$. Notice that

$$\begin{aligned}
\|g'(x)\| = \|\mathbb{1} - F'(x)\| \leq \frac{1}{2}\\
\Rightarrow \|g(x) - g(y)\| \leq \frac{1}{2}\|x-y\| \quad \forall\, x, y \in B
\end{aligned}$$

And so we can see very clearly that g is contracting. Now, since $g(0) = a$, (remember that $F(0) = 0$) we can do the following:

$$\begin{aligned} \|g(x)\| &= \|g(x) - g(0) + a\| \\ &\leq \|g(x) - g(0)\| + \|a\| \\ &\leq \frac{1}{2}\|x\| + \|a\| \\ &\leq \frac{1}{2}r + \frac{1}{2}r \quad = \quad r \end{aligned}$$

Which shows that g maps B into B and is continuous. This means we are ready to construct our useful sequence:

Choose $x_0 \in B$ and set $x_{n+1} = g(x_n)$ converging to, say, $z \in B$

A point of convergence eh? Lets just take a closer look at that:

$$g(z) = g[\lim_{n\to\infty} x_n] = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(x_{n+1}) = z$$

That's nice, out limit point is a fixed point. What a coincidence! This is great though, because now we can say:

$$\begin{aligned} & g(z) = z - F(z) + a && \text{Naturally} \\ & \Rightarrow z = z - F(z) + a && \text{The reason we wanted a fixed point} \\ & \therefore F(z) = a && \text{Obviously} \\ & \therefore a \in F(B) && \text{And since } a \in \frac{1}{2}B\ldots \\ & \therefore \frac{1}{2}B \subset F(B) && \text{Just the way we wanted it.} \end{aligned}$$

All right, now we have all this cool stuff about out lovely function, $F(x)$. What about the space it acts on? If you care to look back to when we stated our theorem (if you can remember that far), you'll see that we want our function to be 1. a diffeomorphism and 2. from *open* W onto *open* $f(W)$. We'll get cracking on the second point, that'll flow into the first and that will lead us nicely to our conclusion. See, we're nearly done. So, without further ado, let's define an open ball:

Let $\frac{1}{2}B_0 = \{x \mid \|x\| < \frac{1}{2}r\}$ be the interior of a Ball at centre a with radius $\frac{1}{2}r$.
While we're at it, let $U = B_0 \cap F^{-1}[\frac{1}{2}B_0]$.
Now U is open since both B_0 and $\frac{1}{2}B_0$ and F is continuous.
This means $F(U)$ must also be open.

Hurray, so we can now see that $F : U \to F(U)$ is a c^r mapping from open U to open $F(U)$.

This is great news, now we just need to show that $F^{-1} : F(U) \to U$ is also a c^r mapping. For this we effectively set up some stuff to create the condition for differentiability.

Let $x, x+h \in F(U)$, $G(x) = y$, $G(x+h) = y + l$ and $F'(y) = S$ say.

Then $F(y+l) = F(y) + S(l) + \phi(l)$ where $\frac{\|\phi(l)\|}{\|l\|} \to 0$ as $\|l\| \to 0$. This being our normal condition for differentiability. Now we sub in x's for $F(y)$'s and solve for l:

$$\begin{aligned} & \Rightarrow x + h = x + S(l) + \phi(l) \\ & \Rightarrow S(l) = h - \phi(l) \\ & \Rightarrow l = S^{-1}(h) - S^{-1}[\phi(l)] \\ & \text{Sub into } G(x+h) = y + l \\ & \text{Giving } G(x+h) = G(x) + S^{-1}(h) - S^{-1}[\phi(l)] \end{aligned}$$

You know what, this looks promising. Let's check if the remainder term does indeed go to zero in the correct manner:

$$\frac{\left\|S^{-1}[\phi(l)]\right\|}{\|h\|} \leq \frac{\left\|S^{-1}\right\| \|\phi(l)\|}{\|h\|} \frac{\|h\|}{\|l\|}$$

And since $\|l\| \leq 2\|h\|$, as $\|h\| \to 0$ so to does $\|l\|$ and thus the whole remainder term expressed above.
$\therefore G$ is differentiable at x and

$$G'(x) = S^{-1} = [F'(y)]^{-1} = [F'(G(x))]^{-1}$$

Now comes the final conclusion. If G is c^s for some $0 \leq s \leq r$ then G' is the composition of the c^s functions G, F' and $[\ldots]^{-1}$ which means that G' is c^{s+1}. G being $c^s \Rightarrow G'$ is c^{s+1} $\forall\, 0 \leq s \leq r$, $\therefore G$ is c^r as required. □