Lecture 2: Topology and Geometry

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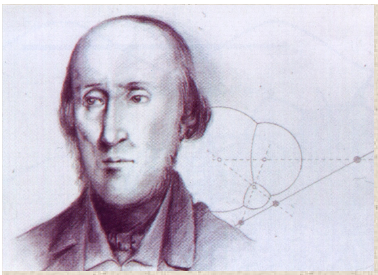
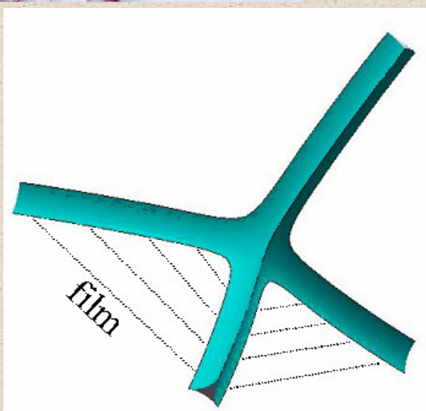
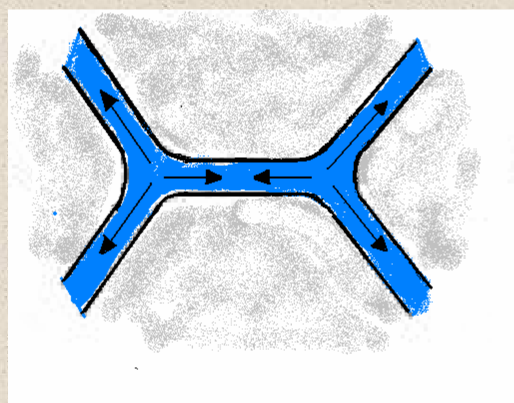
Plan of the lecture

- Plateau rules in 2D and 3D ;
- Relation with topology;
- Minimal surface in 2D: the honeycomb
- Minimal surface in 3D: Kelvin solid, Weaire Phelan ➔ simulations
- Minimal surface : foams are systems in mechanical equilibrium
- Growth laws 2D and 3D



Plateau borders





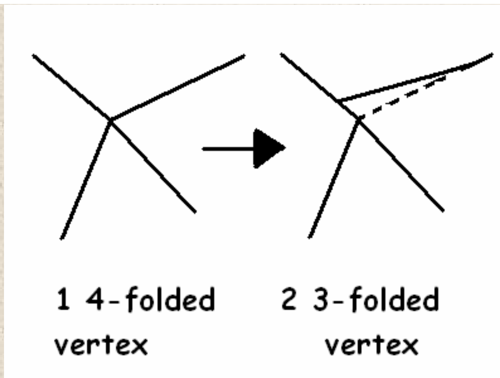
Dry soap films may be thought of as elastic sheets that may branch and are equally stressed all over the froth.

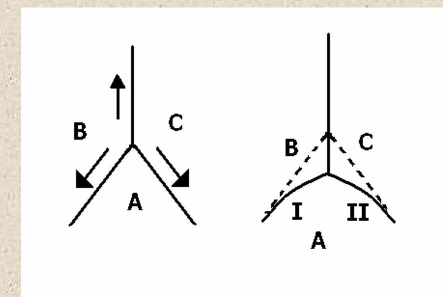
Plateau rules

Equilibrium rule A1: Three films meet at each edge forming 120° angles. ➔ stable equilibrium of three forces.

Equilibrium rule A2: 4 films meet at each vertex (3D), and the angles are all equal.

Equilibrium rule B: The transtion between a Plateau border and an adjacent film is smooth.

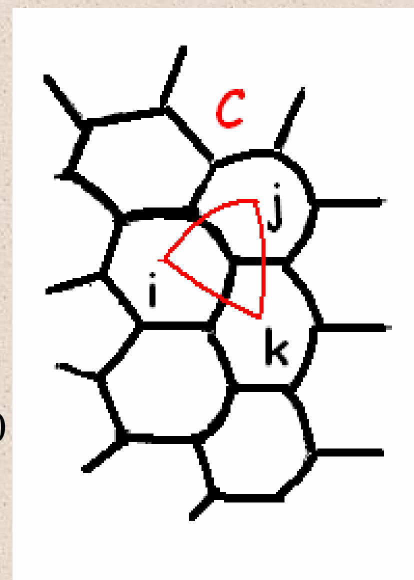




The law of Laplace

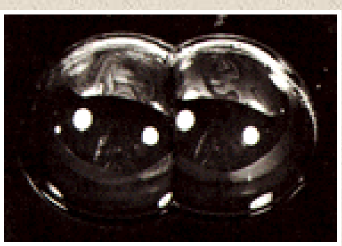
$$\kappa_{ij} = \frac{P_i - P_j}{\gamma}$$

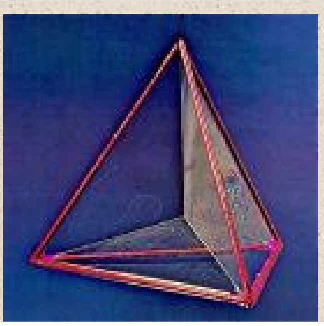
$$\kappa_{ij} + \kappa_{jk} + \kappa_{ki} = \frac{P_i - P_j}{\gamma} + \frac{P_j - P_k}{\gamma} + \frac{P_k - P_i}{\gamma} = 0$$

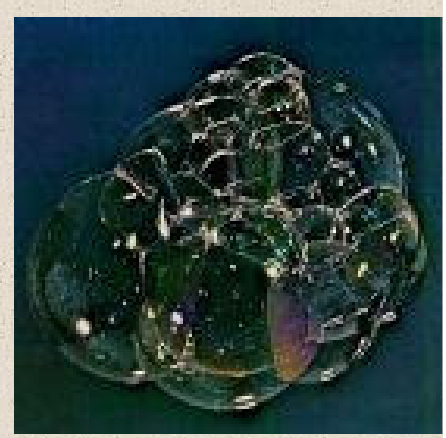


Consequences

- Soap films are in mechanical equilibrium
- Minimize surface➔ Plateau’s problem







Minimal interface tilings: 2D

- What is the arrangement of 2D cells of that tiles 2D space and minimize surface?
- The honeycomb

It is uniform: n=6 for all bubbles

q=0 (topological charge)

No curvature

No pressure difference

J A Glazier and D Weaire

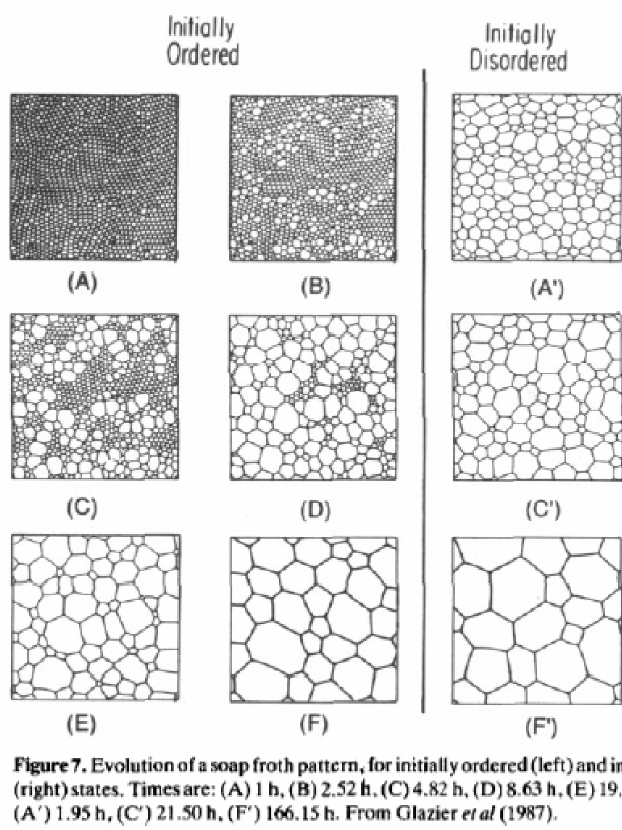


Figure 7. Evolution of a soap froth pattern, for initially ordered (left) and initially disordered (right) states. Times are: (A) 1 h, (B) 2.52 h, (C) 4.82 h, (D) 8.63 h, (E) 19.87 h, (F) 52.33 h; (A') 1.95 h, (C') 21.50 h, (F') 166.15 h. From Glazier *et al* (1987).

Minimal interface tilings 3D

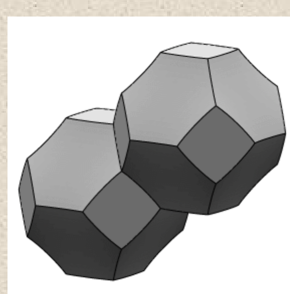
- There is not a polyhedron that tiles 3D spaces with four folded vertices and all angles of 109.47^o.
- Kelvin's problem: what is the arrangement of cells of equal volume that tiles the space minimizing surface area?

- Kelvin' answer : truncated octahedron (tetrakaidecahedron)
- Weaire Phelan cell:

Set of 2 dodecahedra

6 polyhedra with 14 faces: 2 hexagons and 12 pentagons

0.3% less surface



phelanc.fe for the Weaire-Phelan partition.

twointor.fe for Kelvin's partition

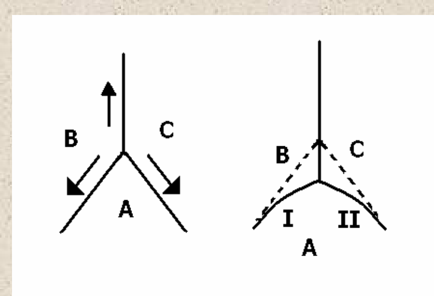
Foam evolution

Plateau rules ➔ walls equilibration very fast!

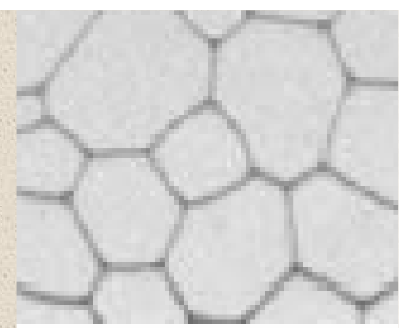
➔ pressure differences

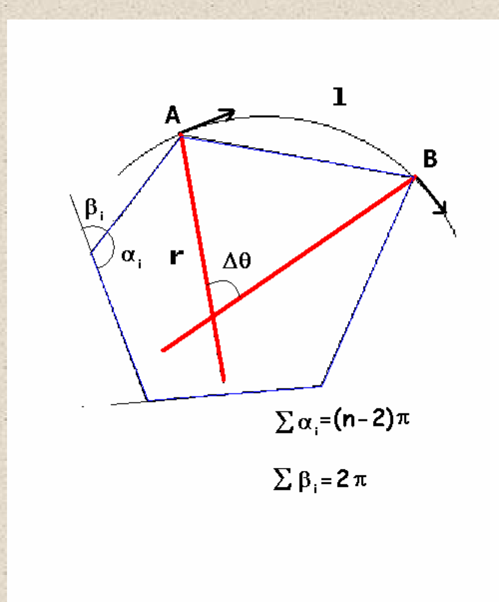
Laplace's Law ➔ gas diffusion

➔ slower



Topological Charge in 2D





120° angles

$$\sum \alpha_i = n\frac{\pi}{3}$$

$$\sum \beta_i = n\pi - \sum \alpha_i = \frac{2n\pi}{3}$$

$$2\pi - \sum \beta_i = \frac{2\pi}{3}(6-n)$$

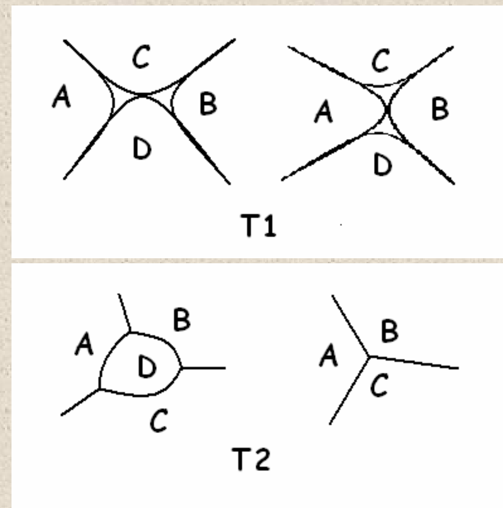
$$q = \frac{\pi}{3}(6-n)$$

Growth law in 2D

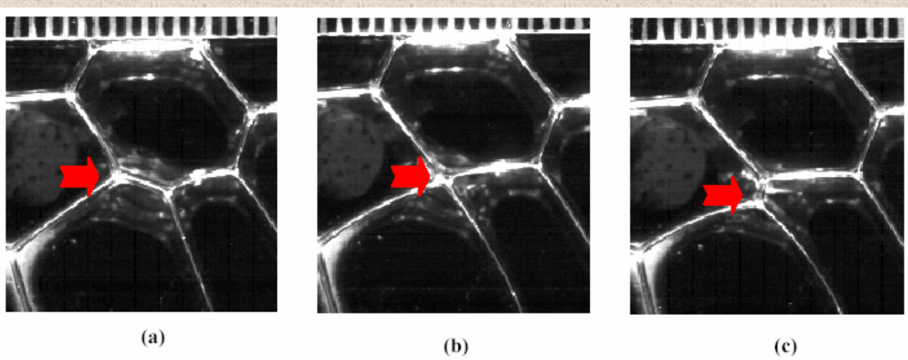
- Von Neumann's law

$$\frac{da}{dt} = k(n-6)$$

- n<6-bubbles shrink ➔ They disappear!
- n>6-bubbles grow
- T1 and T2 arrangements keep <n>=6
- The number of bubbles decrease ➔ the average area of the cells increase



Topological changes: 2D –T1



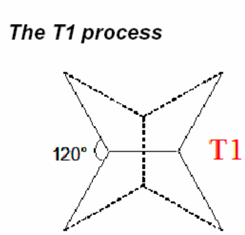


Figure 2: T
dimensional f
the growth of
the minimal p

HARVARD DEAS REU PROGRAM
FINAL REPORT, AUGUST 12, 2003

Timing Film Formation during the T1 Process

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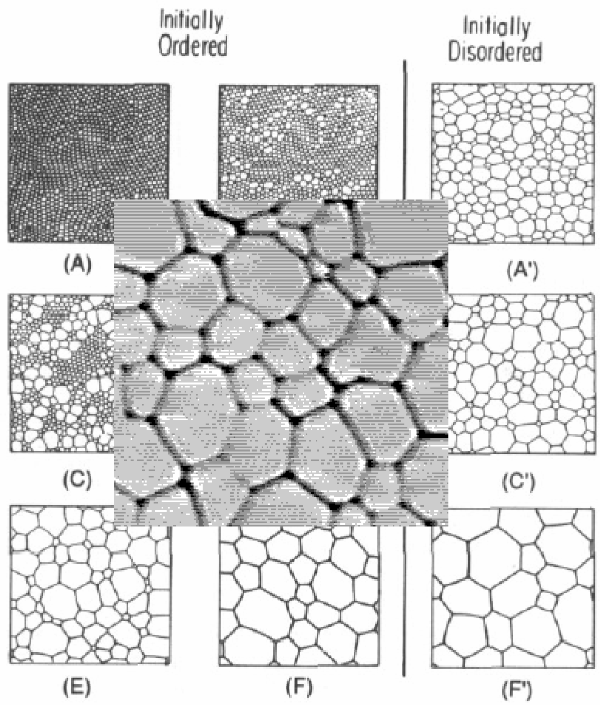
J A Glazier and D Weaire



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Six sided cells:

tile the plane

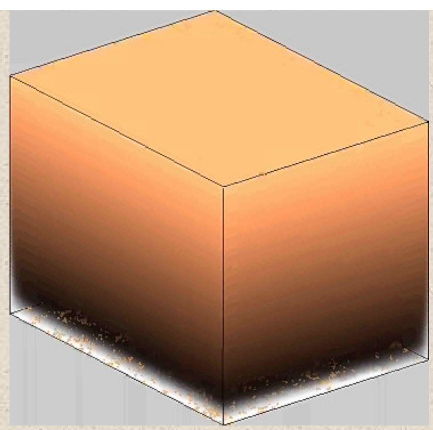
$\langle n \rangle = 6$

$da/dt = 0$

$q = 0$

Growth laws in 3D

- 3D soap froths coarsen.



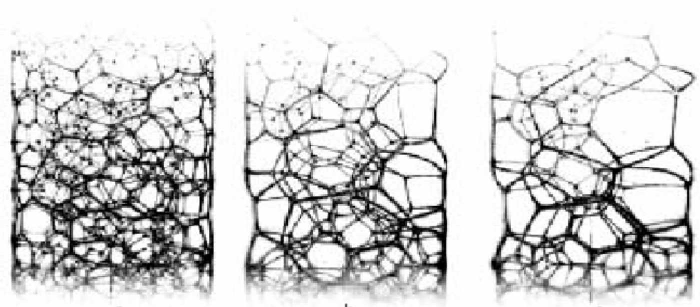
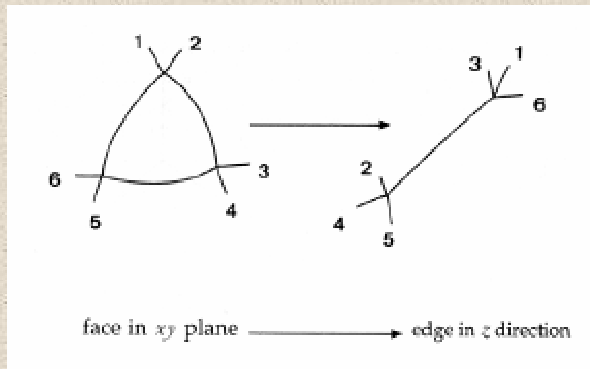
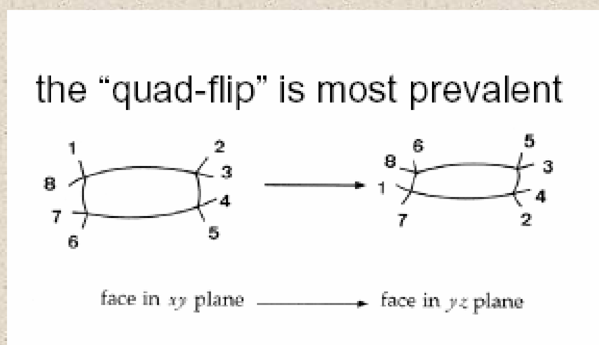


Figure 3. Maximum intensity projections of three-dimensional MRI reconstructions of a foam at three stages of development (a) = 24 hrs. (b) = 36 hrs. (c) = 48 hrs.

Topological Changes: 3D





Growth Law in 3D

$$K_m = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\Delta P \propto \left(\frac{1}{r_1} + \frac{1}{r_2} \right)$$ but $$\iint \frac{1}{r_1 r_2} dS \propto 2\pi - n(\pi - \theta_0)$$

$$\frac{dV}{dt} = \kappa \oiint_S \vec{K}_m \bullet ds$$

$$\frac{dV}{dt} = \kappa \oiint_S \vec{K}_m \bullet ds$$

length^{-1} · surface · volume/time · surface/time

Suppose two bubbles with the same shape, but different sizes. The integral should depend on $V^{-1/3}$ x C(shape). Or

$$\frac{dV^{2/3}}{dt} = \frac{1}{V^{1/3}} \frac{dV}{dt} = C(shape) \Rightarrow C(f)$$

Glazier 1993

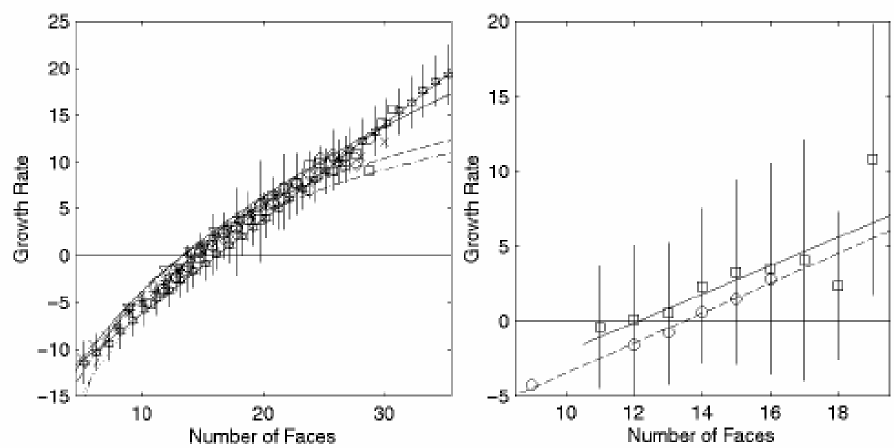


Figure 2a. Growth rates as a function of number of faces for Fortes' edge and vertex models (Dashed lines), Mullins' model (Solid Line), Kawasaki's vertex model (Squares, Circles and Diamonds), Weygand's vertex model (Dotted Line), Monnereau *et al.*'s boundary dynamics model (Triangles), Wakai's boundary dynamics model (Xs and Stars) and Glazier's Potts model (Star of David). 2b. Monnereau *et al.*'s optical tomography experiments (Circles) and our MRI experiments (Squares).

Cellular Potts Model Simulations

$$E = \sum_{\langle ij \rangle} \left(1 - \delta\left(s_i - s_j\right)\right)$$

Sum over farer neighbors reduces anisotropy

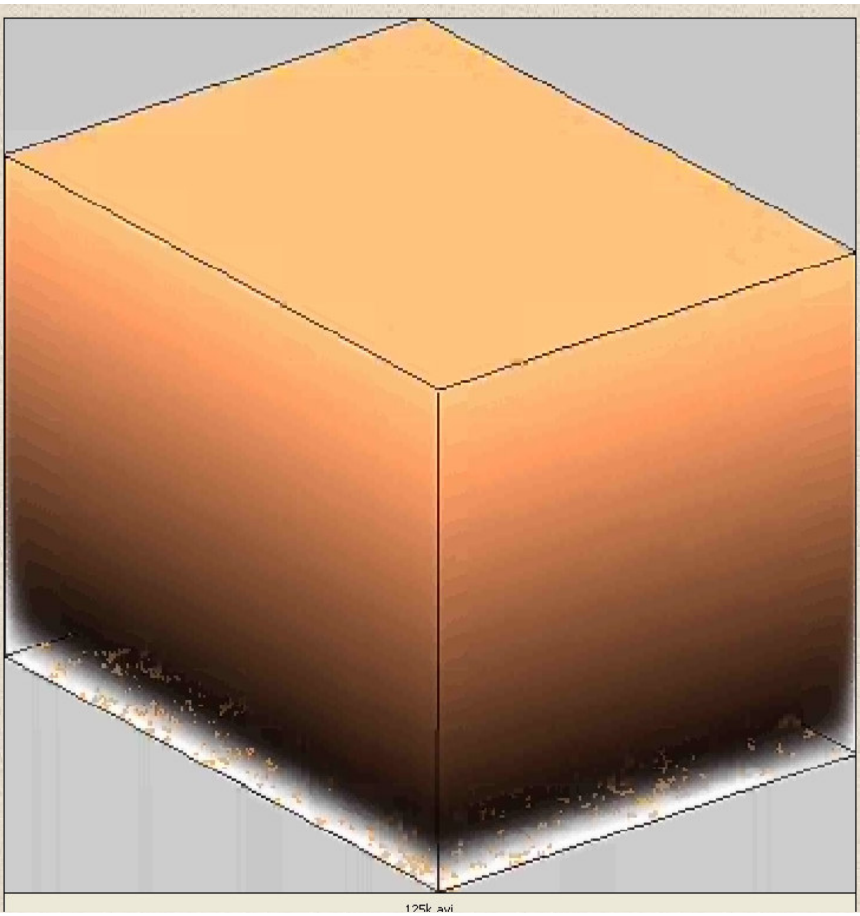
1	1	1	0	0	0	0	0	0	9	8	8	8	8
1	1	2	2	0	0	0	0	9	9	8	8	8	8
1	2	2	2	2	0	0	9	9	9	9	8	8	8
2	2	2	2	2	5	5	5	9	9	9	7	8	8
3	2	2	2	5	5	5	5	5	9	7	7	7	7
3	3	2	5	5	5	5	5	5	5	7	7	7	7
3	3	3	5	5	5	5	5	5	7	7	7	7	7
3	3	4	4	5	5	5	5	5	7	7	7	7	7
3	4	4	4	4	5	5	5	6	6	7	7	7	7
4	4	4	4	4	5	5	6	6	6	6	7	7	7

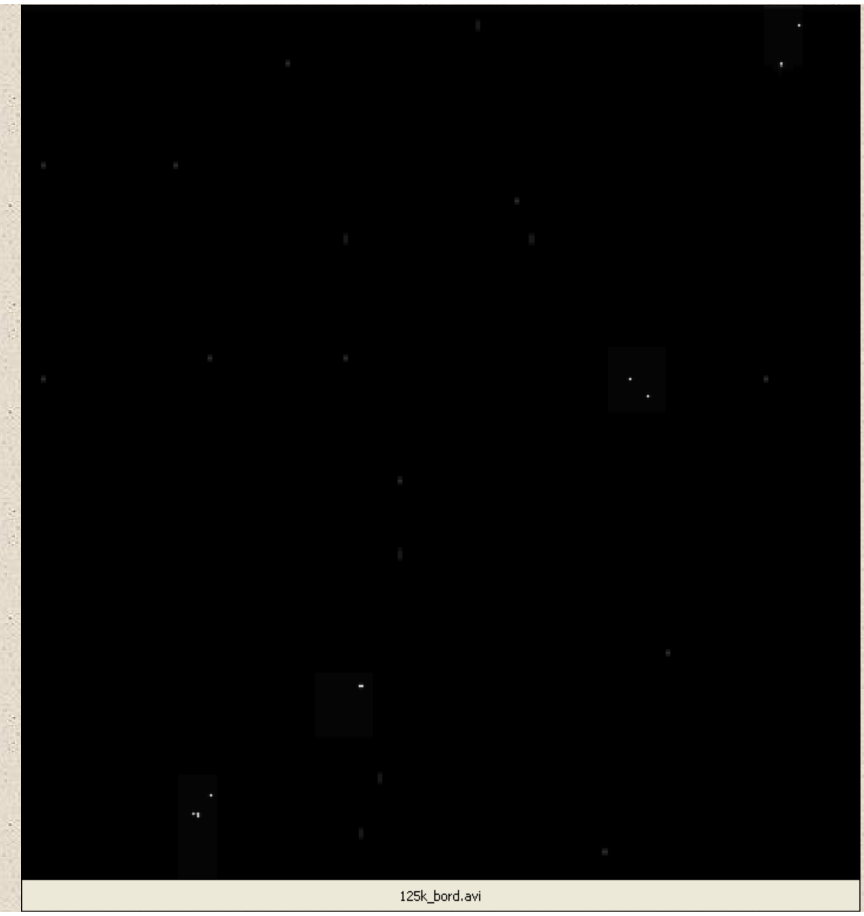
The simulations

- 3D square lattice with $L_x x L_y x L_z$ sites.
- Periodic boundary conditions.
- Initial conditions:
  Cubish cells with a given size.
  Target size distribution predefined.
  Pre-relaxation considering target volume energy term ($\sim(v-v_t)^2$)

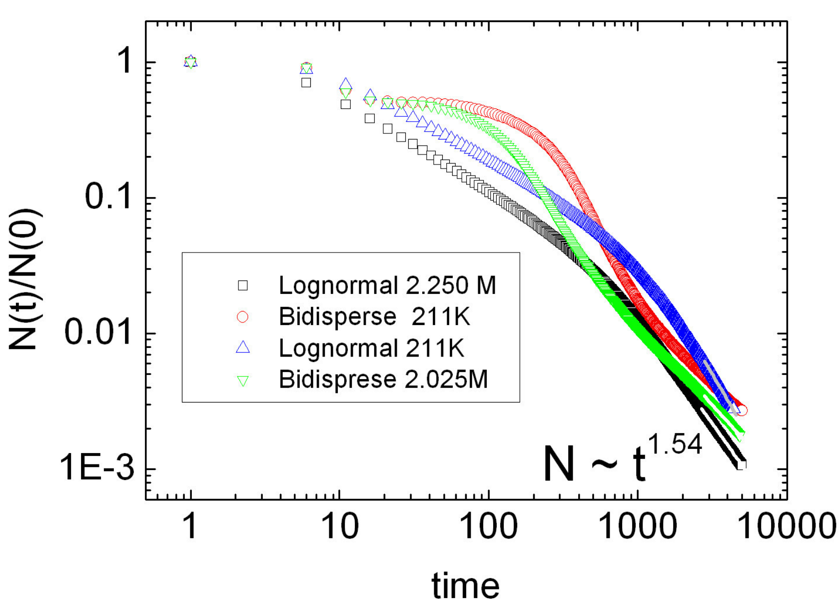
The dynamics

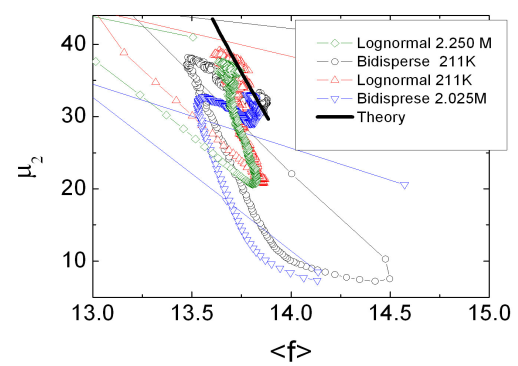
- Choose a site i and a first neighbor j.
- If $s_i \neq s_j$ virtually put $s_i = s_j$ and calculate energy of the virtual configuration.
- If the energy is lowered accept the change.
- 1 MCS = number of choices equal to the number of sites.



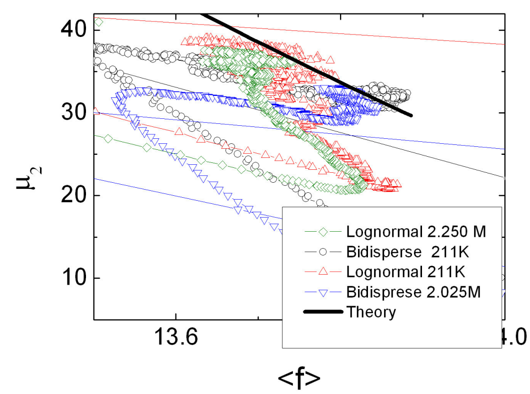


Time evolution

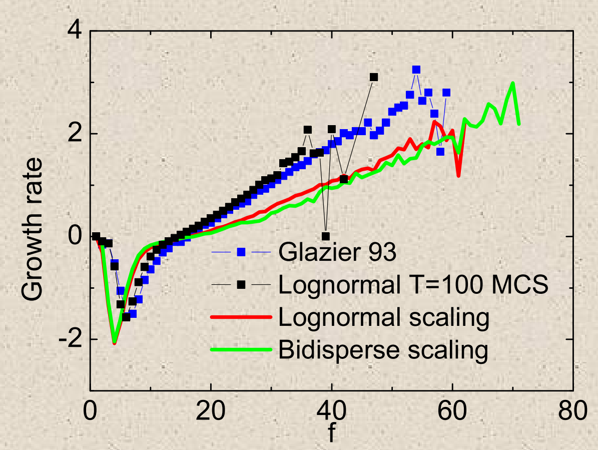




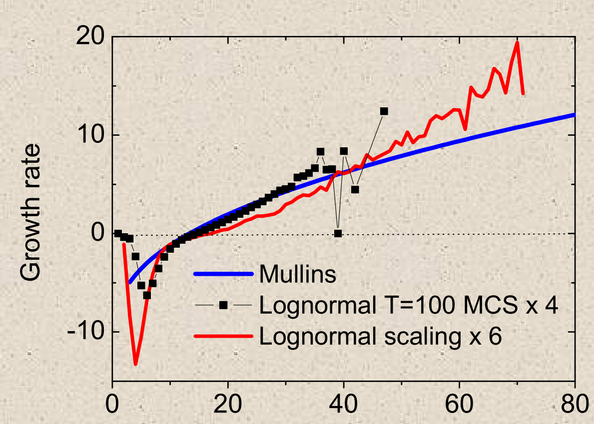
Lognormal 2.250 M
Bidisperse 211K
Lognormal 211K
Bidisprese 2.025M
Theory
μ_2
<f>



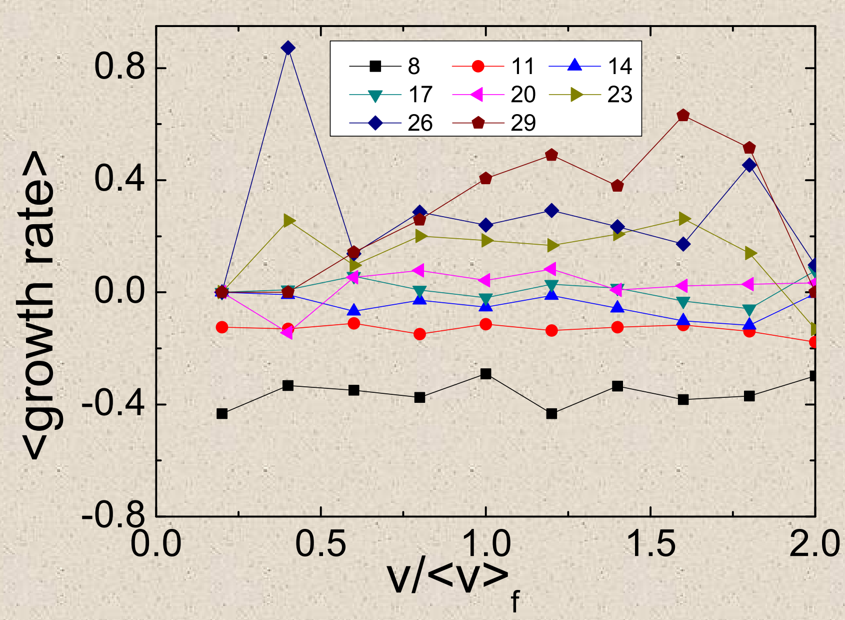
Lognormal 2.250 M
Bidisperse 211K
Lognormal 211K
Bidisprese 2.025M
Theory
μ_2
<f>



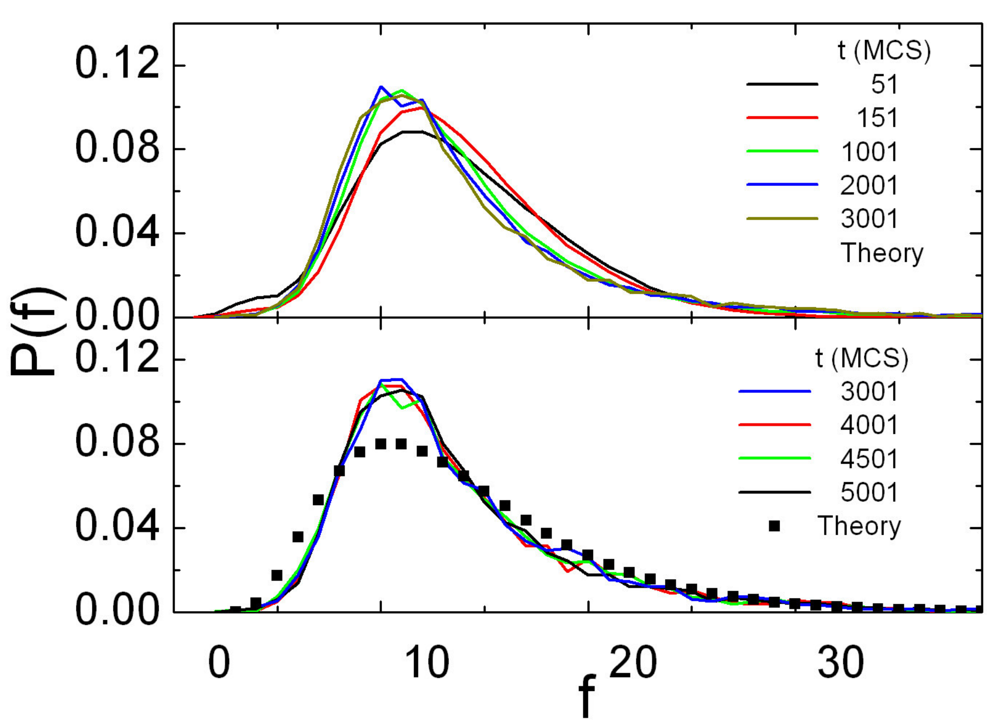
Growth rate
Glazier 93
Lognormal T=100 MCS
Lognormal scaling
Bidisperse scaling
f



Growth rate
Mullins
Lognormal T=100 MCS x 4
Lognormal scaling x 6



<growth rate>
0.8
0.4
0.0
-0.4
-0.8
0.0
0.5
1.0
1.5
2.0
$v/\langle v \rangle_f$
8
11
14
17
20
23
26
29



P(f)
0.12
0.08
0.04
0.00
t (MCS)
51
151
1001
2001
3001
Theory
t (MCS)
3001
4001
4501
5001
Theory
0
10
20
30
f

Next talk

- Topology, geometry, and coarsening:
  Theory and simulations