Lecture 1: Topology

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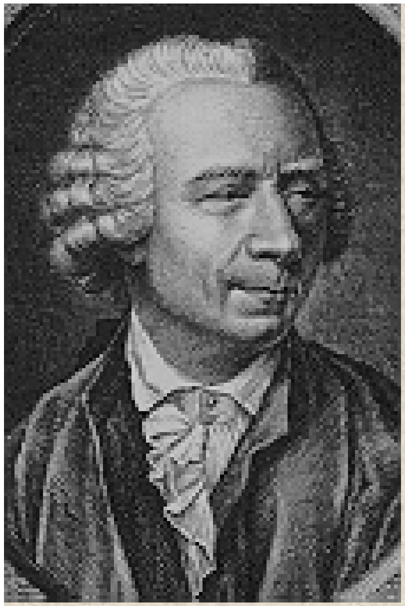
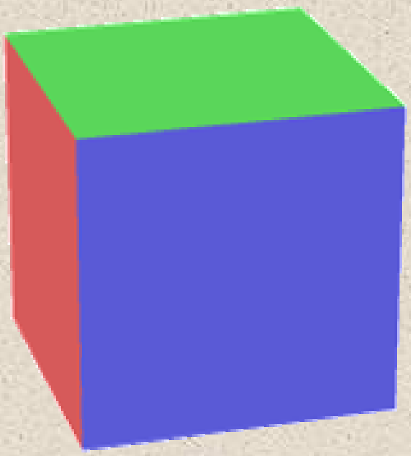
Plan of the lecture

- Euler rules
- Topological charge in 2D
- Topological Gauss Theorem in 2D
- Foam topology: how to characterize?
- Experiments: From Matzke to now what do we measure?
- Average topological quantities
- Measurable topological quantities
- Topological Distribution functions
- Maximum Entropy models (possibly reached in coarsening, see next lecture)



Euler's Rules for convex polyhedra

$$V - E + F = 2$$

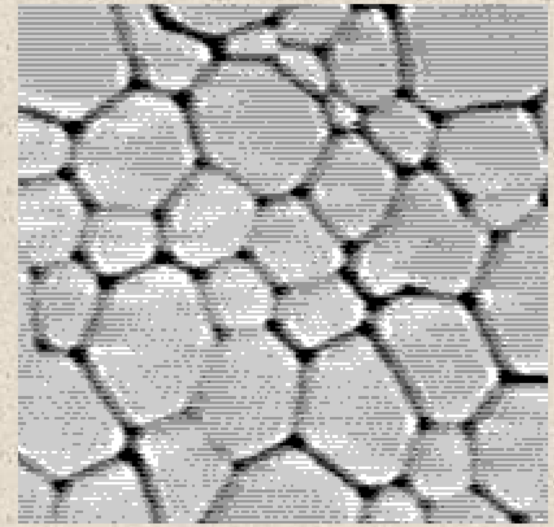


V=8 E=12 F=6

V=4F/3 E=4F/2

V=12 E= 30 F=20

V=3F/5 E=3F/2



Euler's Rules for a two dimensional tiling: 1 polyhedron, F $\to\infty$

$$2V = 3E$$

$$V - E + F = 2$$

$$\frac{2}{3}\frac{E}{F} - \frac{E}{F} + 1 = \frac{2}{F}$$

$$\frac{E}{F} = \frac{\langle n \rangle}{2} = 3$$

$$\langle n \rangle = 6$$

Why three folded vertices

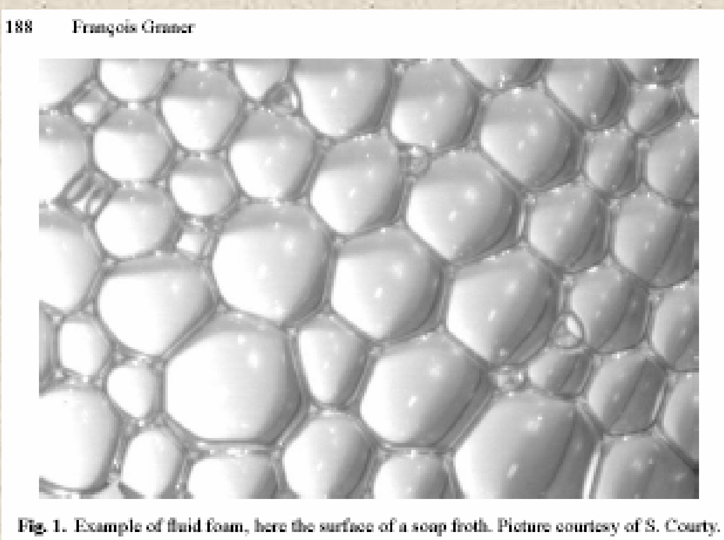

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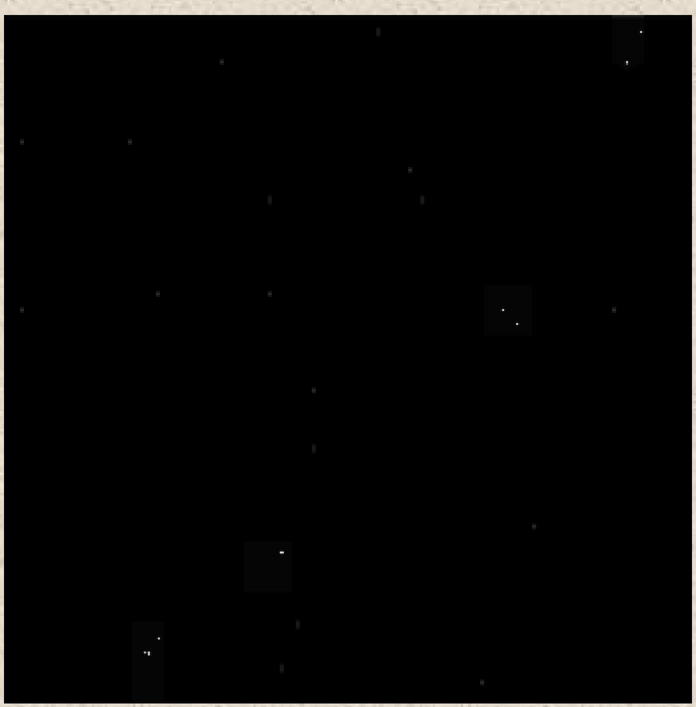
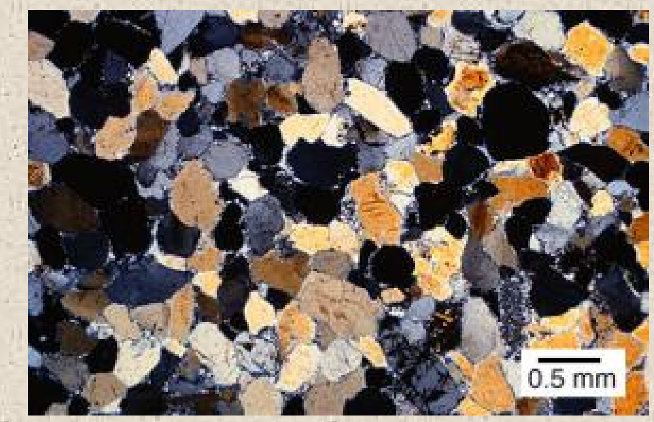
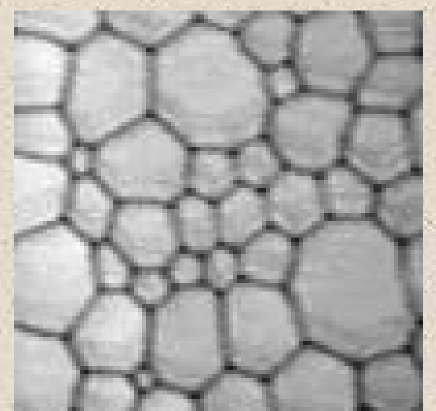
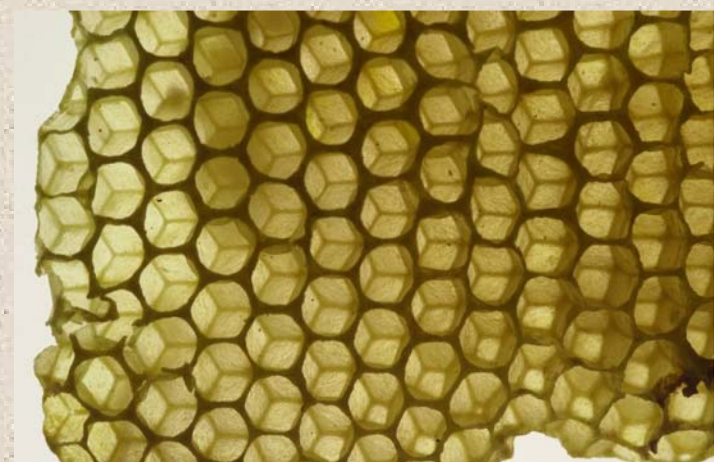
Fig. 1. Example of fluid foam, here the surface of a soap froth. Picture courtesy of S. Courty.







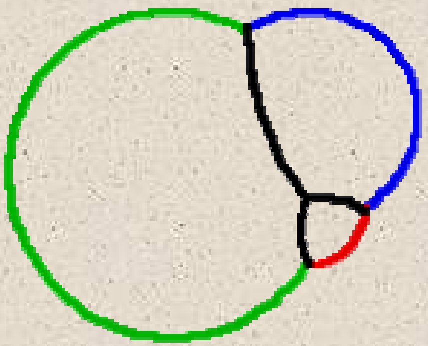


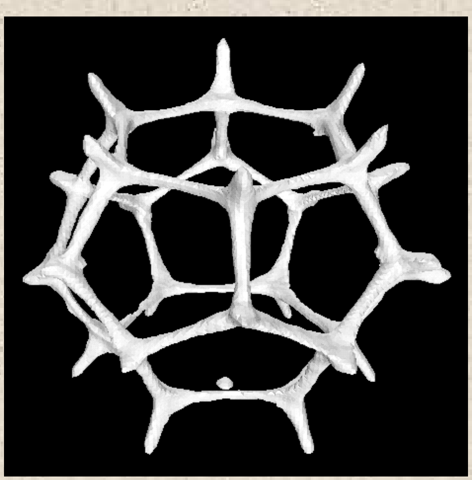
Euler's Rules for tiling 3D spaces

$$V - E + F = 2$$

Apply for isolated bubbles:

3V=2E





Coxeter identity

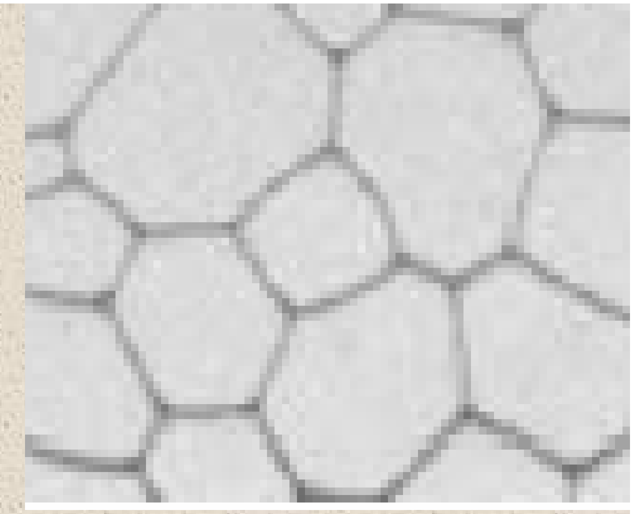
- 3v=2e for each, isolated bubble.
- v-e+f=2 ➔ 3f-e=6
- e is the number of edges of a bubble
- in a bubble each edge belongs to 2 faces
- the average number <n> of *edges per face* is <n>= 2e/f;
- 3f-e=6 ➔ 6f-<n>f=12

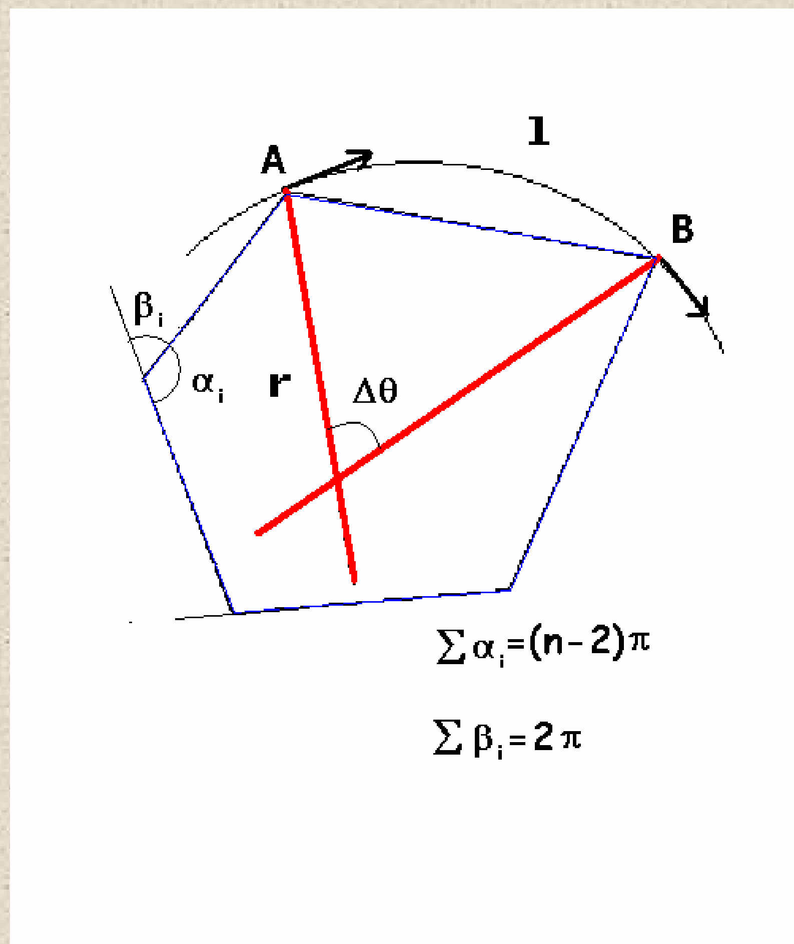
$$f = \frac{12}{< n > -6}$$

Valid for each bubble

Relation between two quantities

Topological Charge in 2D





120° angles

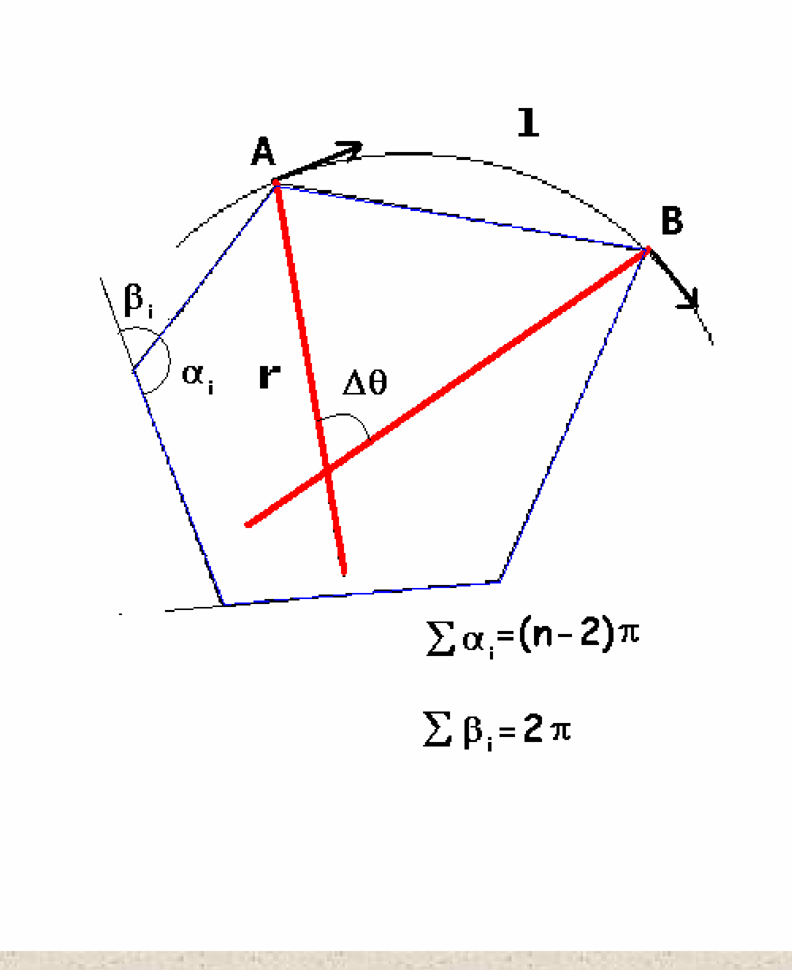
$$\sum \alpha_i = n\frac{\pi}{3}$$

$$\sum \beta_i = n\pi - \sum \alpha_i = \frac{2n\pi}{3}$$

$$2\pi - \sum \beta_i = \frac{2\pi}{3}(6-n)$$

$$q = 2\pi - n\frac{\pi}{3}$$

Gauss Theorem in 2D

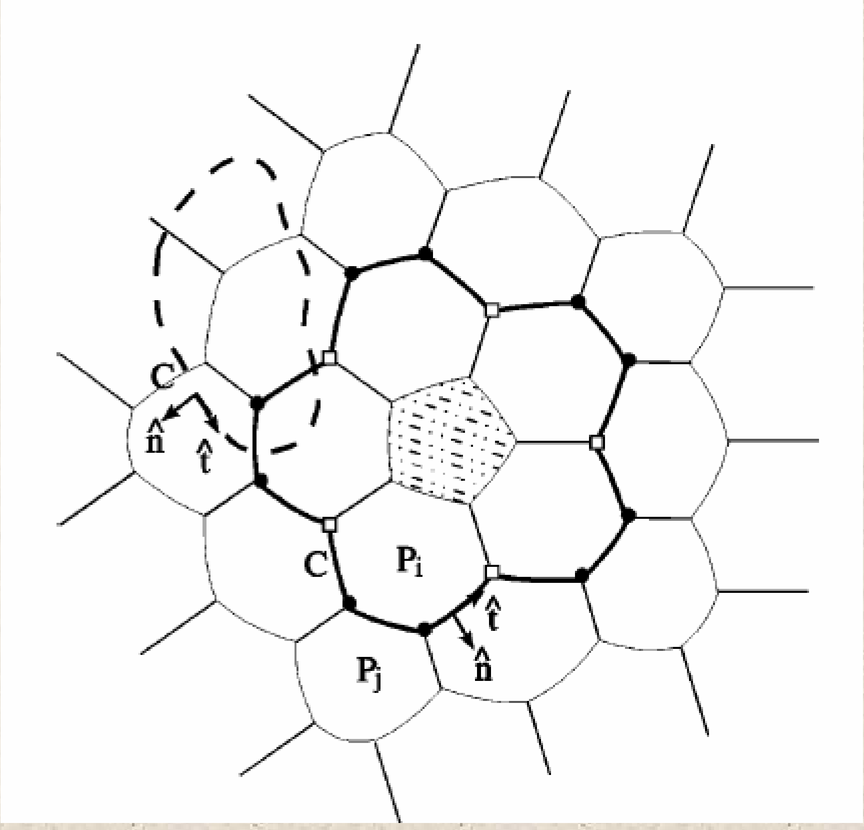


$$\Delta\theta_i = \frac{l_i}{r_i} \qquad \kappa_i = \frac{1}{r_i}$$

$$\sum_i \frac{l_i}{r_i} + n\frac{\pi}{3} = 2\pi$$

$$\sum_i \kappa_i l_i = 2\pi - n\frac{\pi}{3} = q$$

Other contours



$$Q(C) = \sum_{k \in C} q_k = \frac{\pi}{3}\left(6 - v^+ + v^-\right)$$

v^+: outward vertices

v^-: inward vertices

$$\sum_C \kappa_{ij} \ell_{ij} = \sum_{k \in C} q_k = Q(C)$$

Extending to 3D

- Consider a face of n edges.

$$\iint_S \mathrm{K}\,dS = 2\pi - n\left(\pi - \theta_0\right)$$

where θ_0=109.47º and K is the Gaussian curvature:

$$K = \frac{1}{r_1 r_2}$$

Why is it important?

- 2D: The law of Laplace gives the curvature in terms of pressure difference

- 3D: ??

The law of Laplace gives the mean curvature K_m in terms of pressure difference!

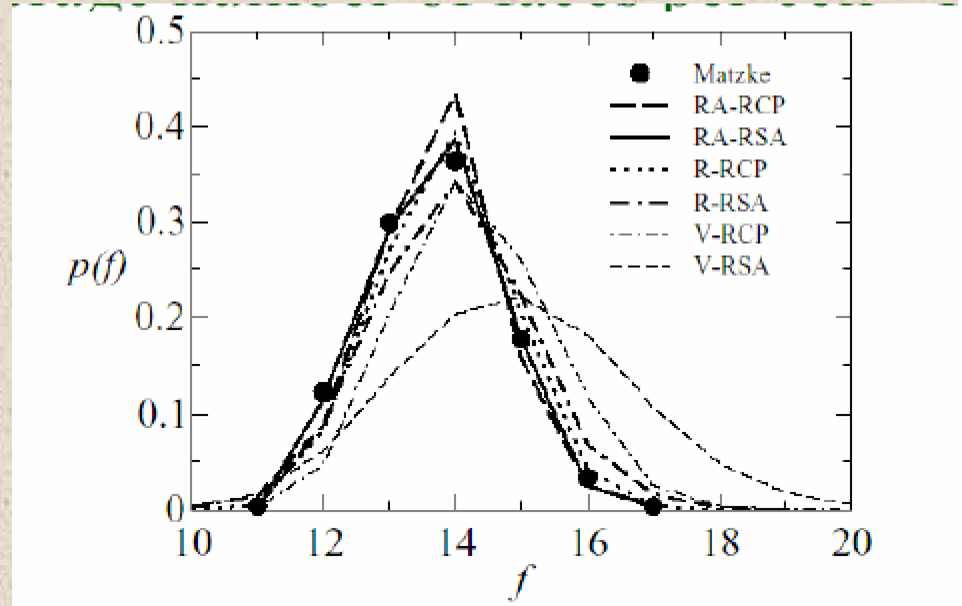
$$K_m = \frac{1}{r_1} + \frac{1}{r_2}$$

Measuring: Matzke

- E.Matzke, *American Journal of Botany 33, 58-80 (1945)*)
- E Matzke and J Nestler *(American Journal of Botany 33, 130-144 (1946))*
- Using a serynge put thousands of monodisperse or bidisperse bubbles by hand.
- Using a binocular stereoscope observed and measured the system. Photographs and drawings.

Matzke Results:

- 600 bubbles in the bulk: <f>~13.70
- 400 bubbles in the boundary: <f>~11.0
- No exception to Plateau's rules.
- No Kelvin cell



- More frequent cells :13 faces

4 faces:1, 5 faces:10 ,

6 faces:2.

- Faces with 4,5 ou 6 edges

Direct images and optical tomography in 2D

- In two dimensions the foam is put between two plexiglass plates: Films, photographs, etc
- Good for coarsening investigation.

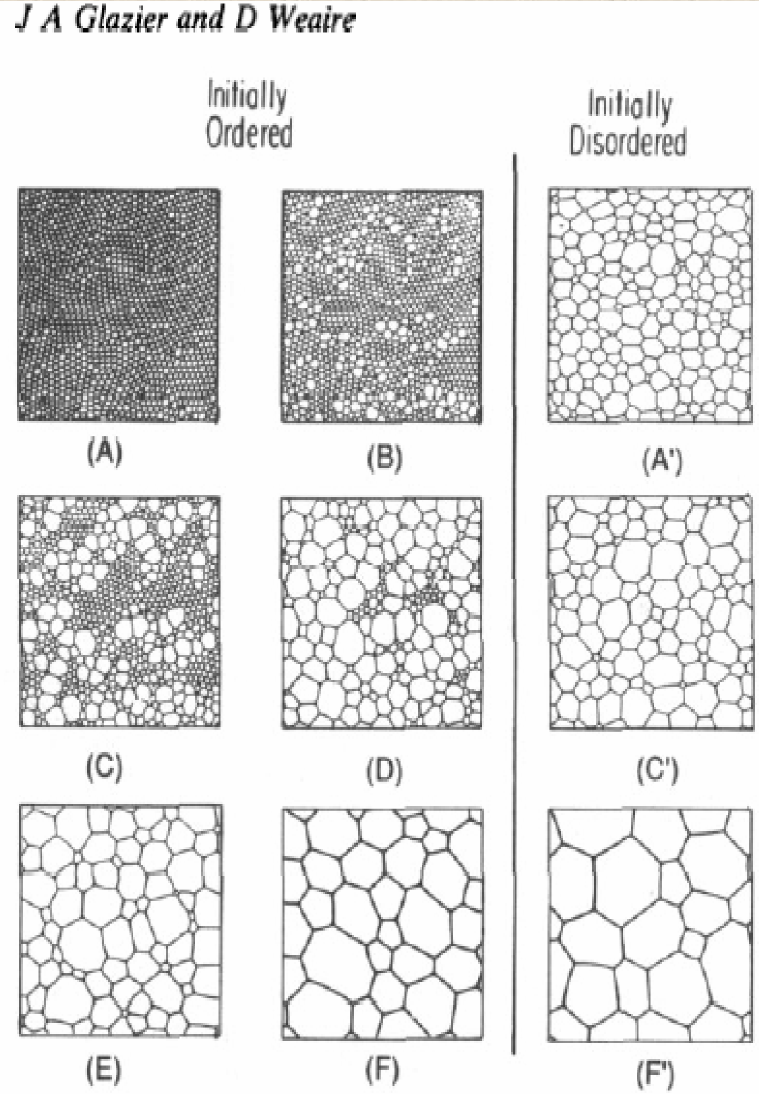
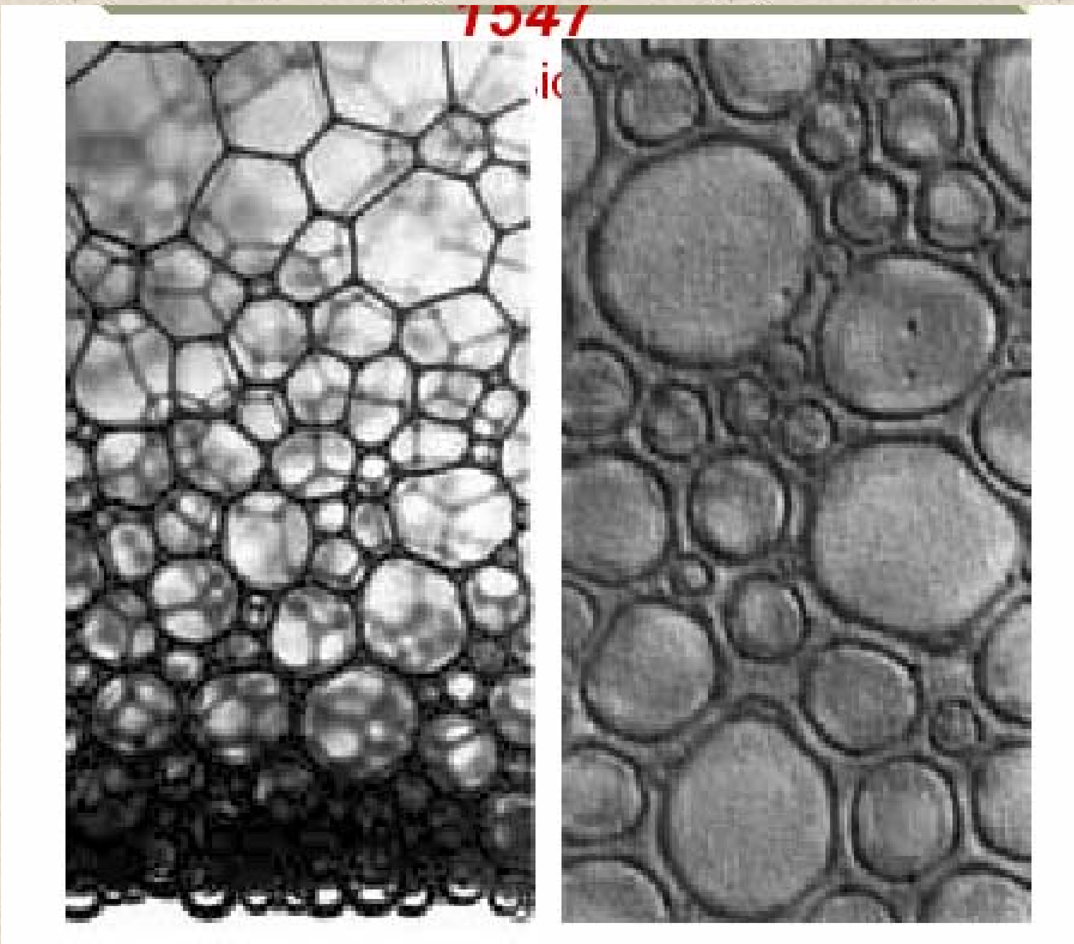


Figure 7. Evolution of a soap froth pattern, for initially ordered (left) and initially disordered (right) states. Times are: (A) 1 h, (B) 2.52 h, (C) 4.82 h, (D) 8.63 h, (E) 19.87 h, (F) 52.33 h; (A') 1.95 h, (C') 21.50 h, (F') 166.15 h. From Glazier *et al* (1987).

What about 3D?

- To look inside a froth it must be very dry!
- Light scattering,then?
- Dry cases: how to deal with so much information?
- Numerical treatment.



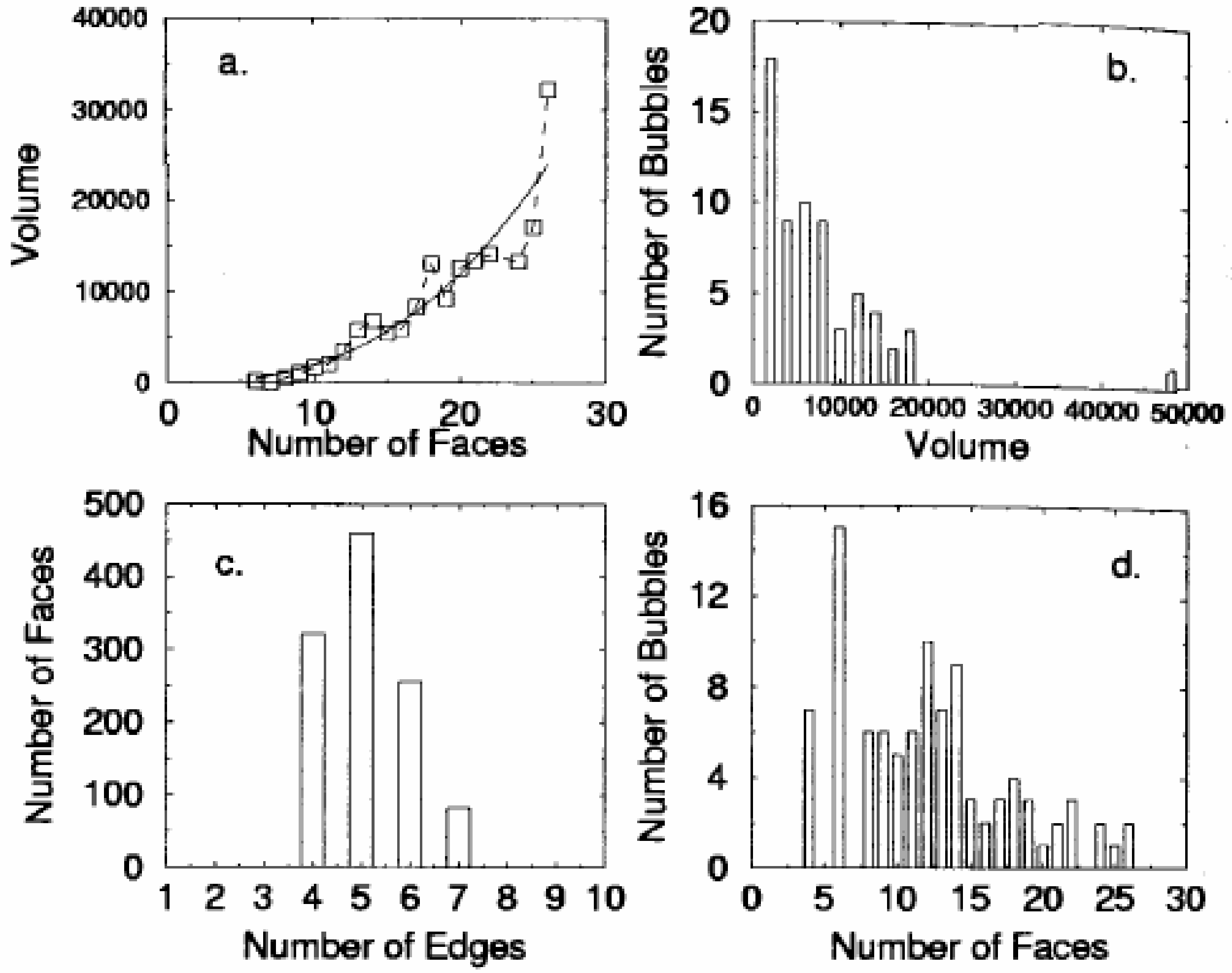


FIG. 2. Detailed structural information from a single data run for a dry foam ($\Phi < 3\%$) at t = 36 hrs. (a) $\langle V_f \rangle$ vs. f. The exponent $\alpha = 2.7$. Distributions of: (b) Volumes. (c) Edges. (d) Faces. Distributions (b) and (d) are wider than at early times, when the foam is much more ordered.

Magnetic Resonance Imaging of Structure and Coarsening in Three-Dimensional Foams

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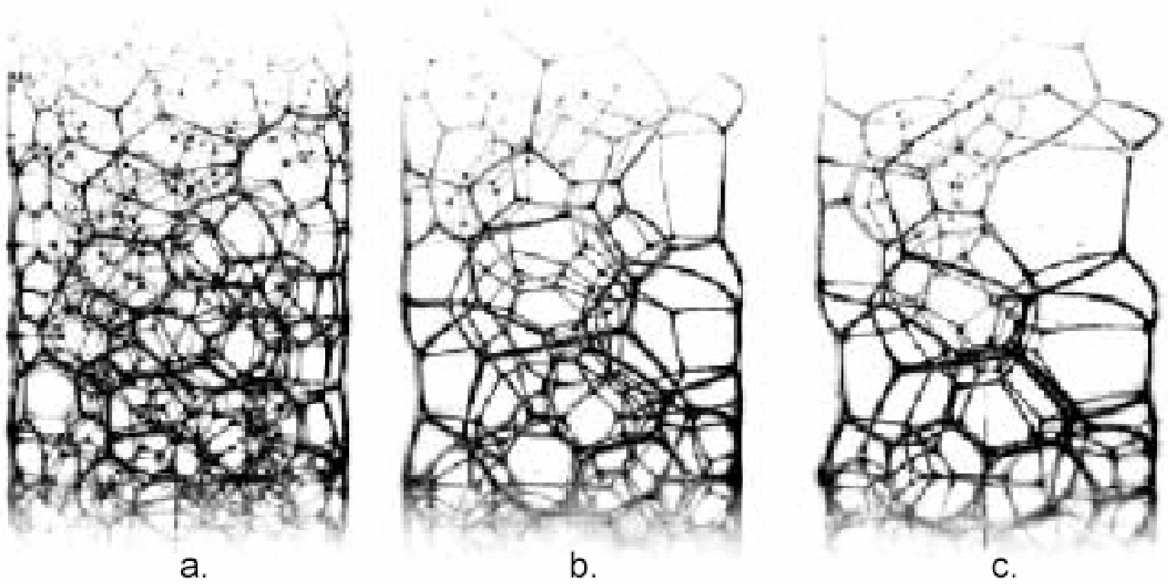


Figure 3. Maximum intensity projections of three-dimensional MRI reconstructions of a foam at three stages of development (a) = 24 hrs. (b) = 36 hrs. (c) = 48 hrs.

Statistics

- What variables define the state of a froth?
- Vertices, edges and faces position.
- 3D: Center of mass position, volume, surface area, number of faces, number and length of edges...and correlations.
- 2D: Center of mass position, area, perimeter, number of edges...and correlations!

Statistics

$$\int dV_b \, p(V_b) = 1$$

$$\int dV_b \, p(V_b) V_b = \overline{V_b} = \frac{1 - \Phi_l}{N_V}$$

$$\rho\left(\frac{V_b}{\overline{V_b}}\right) \frac{dV_b}{\overline{V_b}} = p(V_b) \, dV_b$$

$$\rho\left(\frac{V_b}{\overline{V_b}}\right) = \overline{V_b} \, p(V_b)$$

$$\int dS \, p(S) S = \overline{S}$$

$$\rho\left(\frac{S}{\overline{S}}\right) = \overline{S} \, p(S)$$

$$\sum p(F) = 1$$

$$\sum p(F) F = \overline{F}$$

Statistics: 2D

$$\int dA_b \, p(A_b) A_b = \overline{A}$$

$$\rho\left(\frac{A_b}{\overline{A_b}}\right) = \overline{A}_b \, p(A_b)$$

$$\sum p(n) = 1$$

$$\sum p(n) = \overline{n} = 6$$

μ_2 and m(n)

$$\mu_2 = \sum_n \left(n - \bar{n}\right)^2 p(n) \qquad \mu_2 = \sum_n \left(f - \bar{f}\right)^2 p(f)$$

Aboav-Weaire law

$$m(n) = A + \frac{B}{n} \qquad m(f) = A + \frac{B}{f}$$

$$\sum_n m(n)\, n\ p(n) = \sum_n n^2 p(n)$$ identity

Compatible forms

$$m(n) = 5 + \frac{6 + \mu_2}{n} \qquad m(n) = 6 - a + \frac{6a + \mu_2}{n}$$

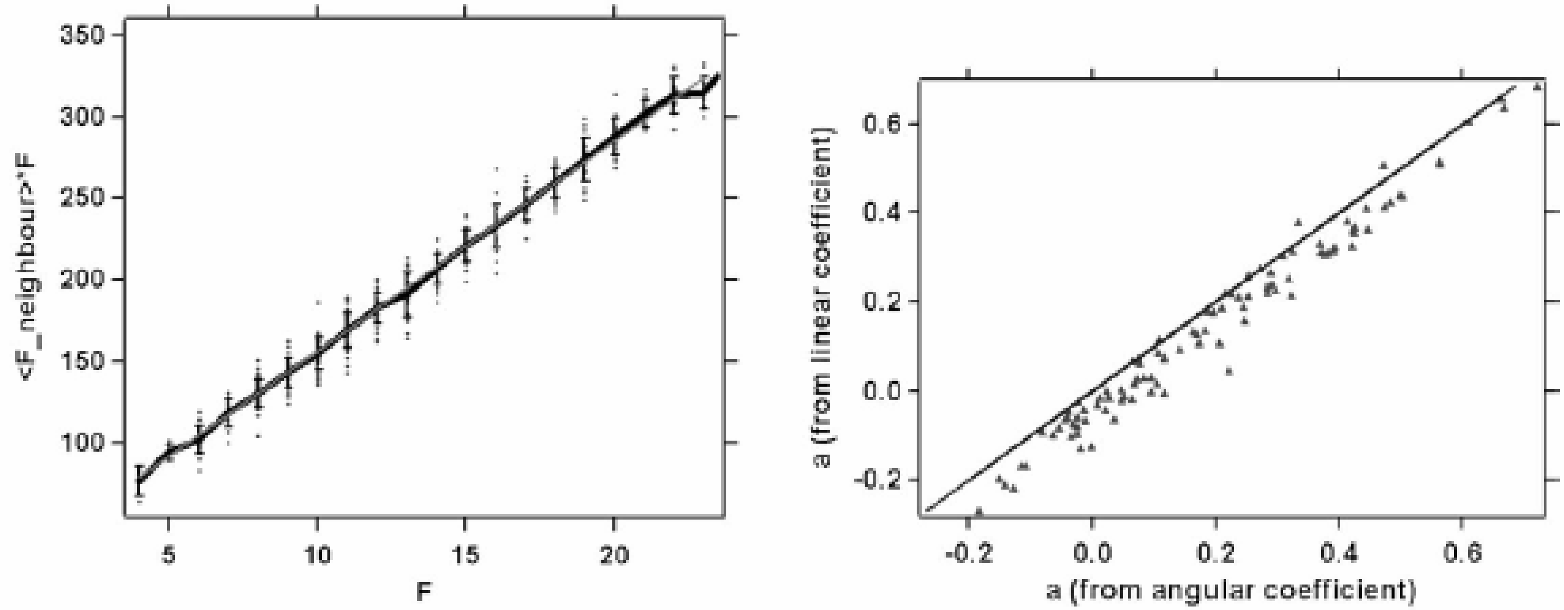
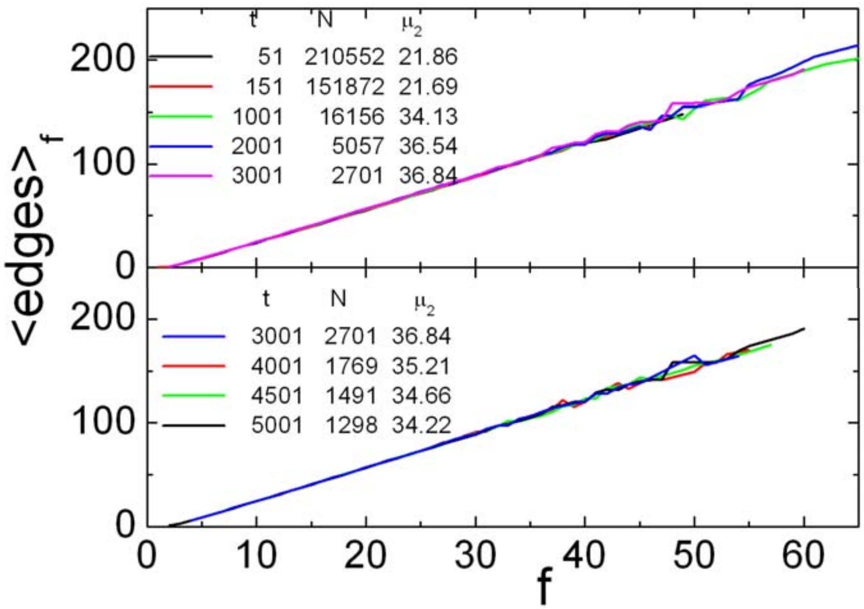


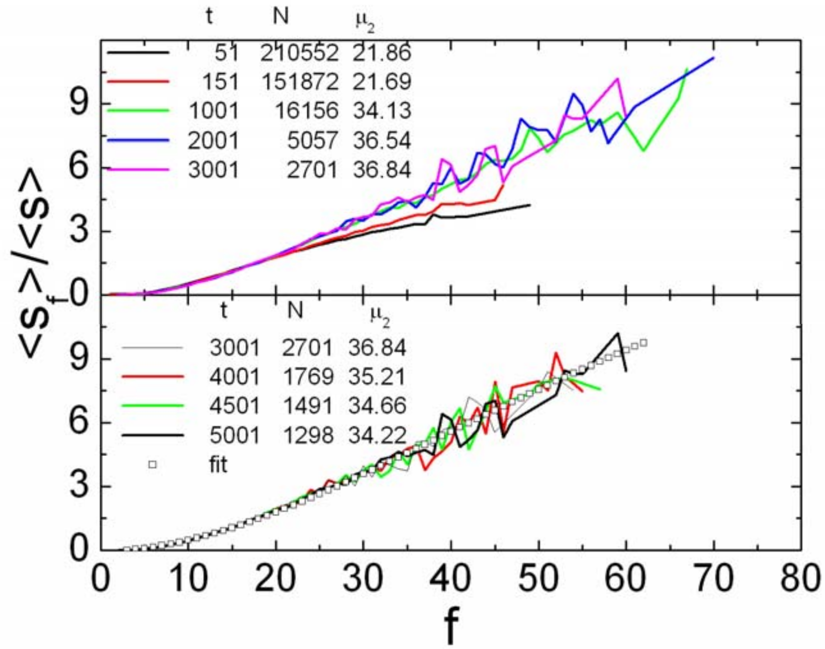
Fig. 5. 3D Aboav–Weaire law. (Left) Correlation between the number of faces of a bubble and of its neighbours for a coarsening foam at a given time; dots = individual bubble values; solid line = average for each value of F; bars = standard deviation for each value of F; thin grey line = linear fit. (Right) Two different measures (see text) of the unknown parameter a from the linear fit by assuming Eq. (7); one is plotted vs. the other for the same foam at each time-step. The solid line is the diagonal ($y = x$).

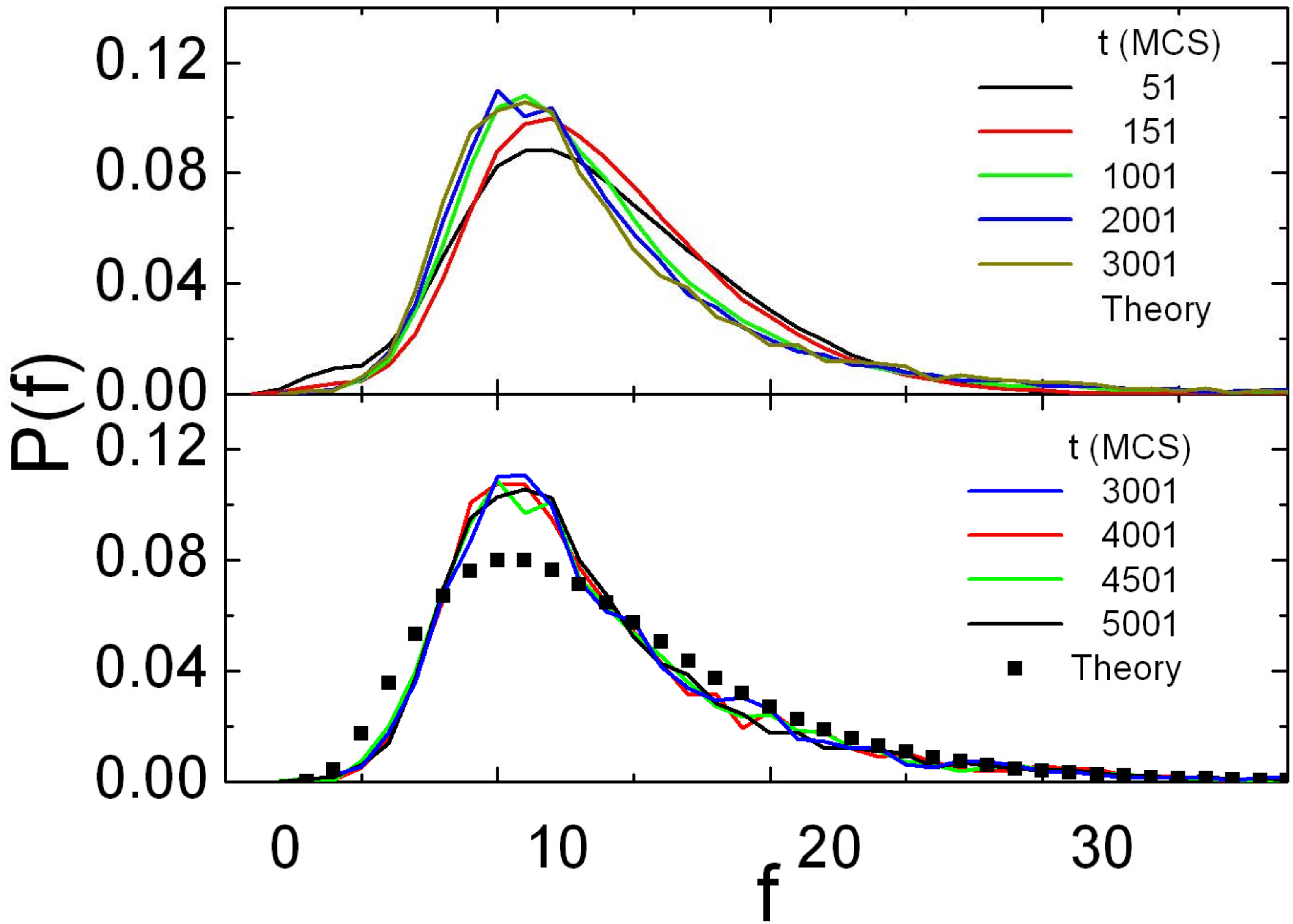
Dry three-dimensional bubbles: growth-rate, scaling state and correlations

Stéphane Jurine[a], Simon Cox[b,1], François Graner[a,*]

Colloids and Surfaces A: Physicochem. Eng. Aspects 263 (2005) 18–26


<edges>$_f$
t N μ_2
51 210552 21.86
151 151872 21.69
1001 16156 34.13
2001 5057 36.54
3001 2701 36.84
t N μ_2
3001 2701 36.84
4001 1769 35.21
4501 1491 34.66
5001 1298 34.22
f


<s$_f$>/<s>
t N μ_2
51 210552 21.86
151 151872 21.69
1001 16156 34.13
2001 5057 36.54
3001 2701 36.84
t N μ_2
3001 2701 36.84
4001 1769 35.21
4501 1491 34.66
5001 1298 34.22
fit
f



P(f)
0.12
0.08
0.04
0.00
t (MCS)
51
151
1001
2001
3001
Theory
0.12
0.08
0.04
0.00
t (MCS)
3001
4001
4501
5001
Theory
0
10
20
30
f

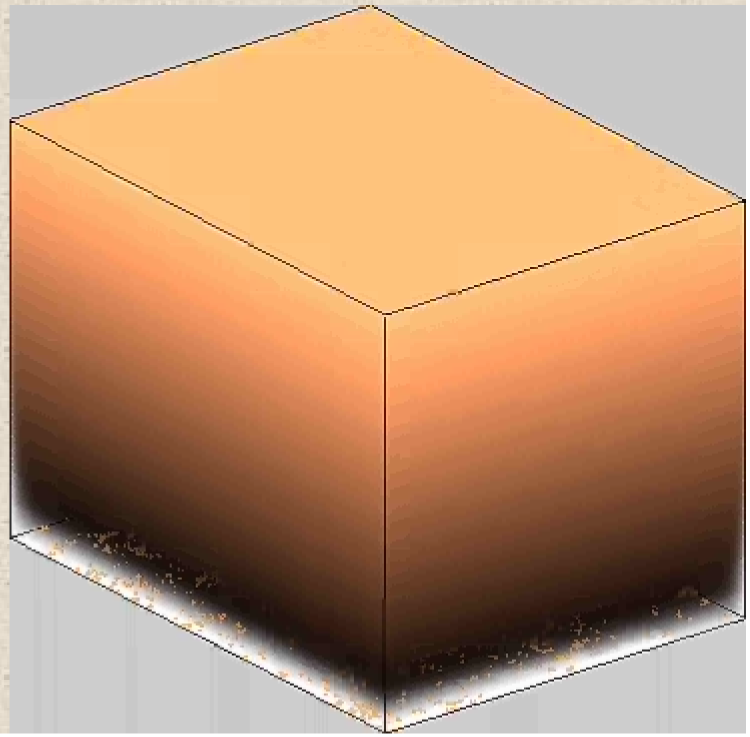
Time evolution

- Foams certainly evolve in time.
- Dynamics reflects in bubbles shapes.
- It is worth simulating!
- Is there a scale invariant regime?

  Distributions of non-dimensional quantities should not vary in time.

Yes in 2D

?? It seems yes in 3D.



Next talks

- Topology and geometry
- Topology, geometry, and coarsening