The Foam Drainage Equation

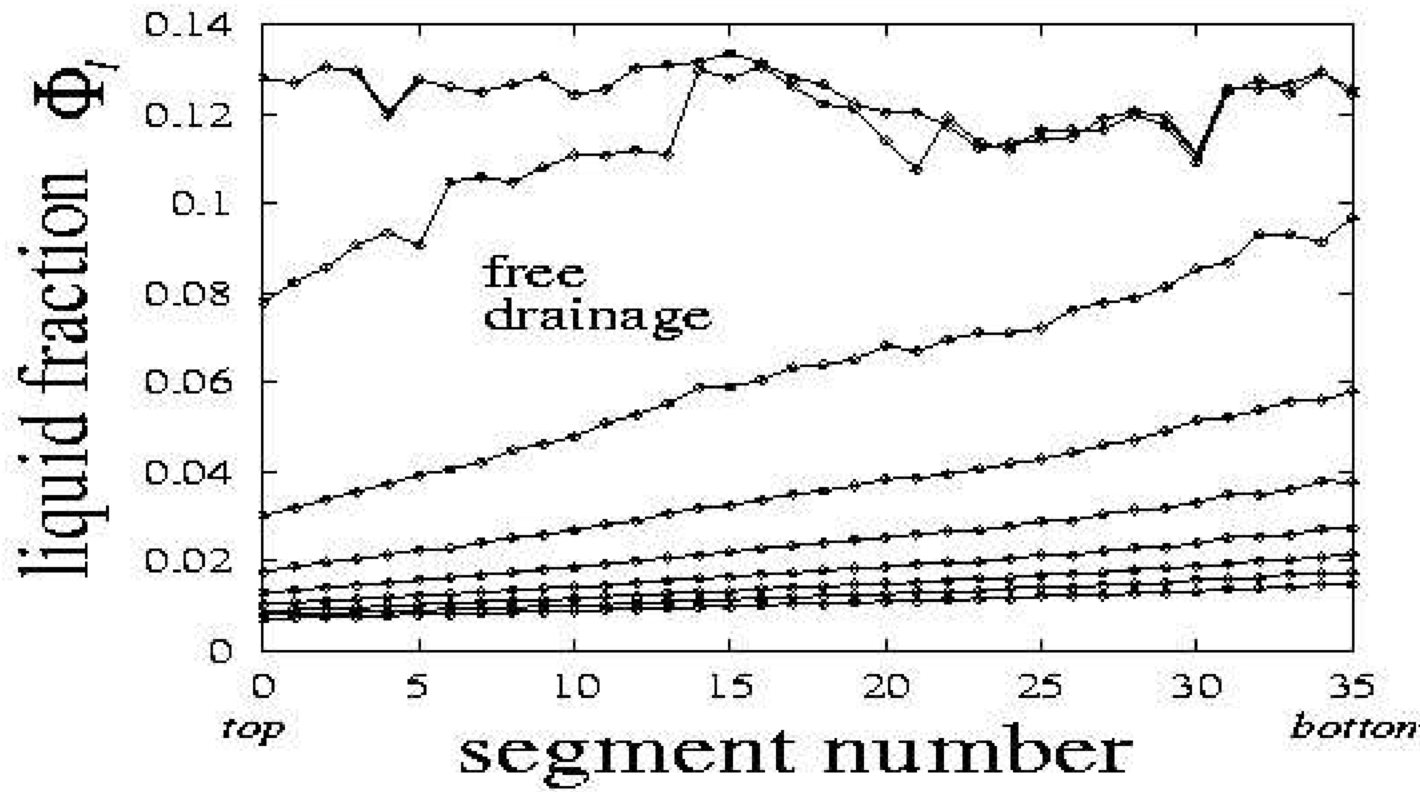
Simon Cox

foams@aber.ac.uk
Institute of Mathematical and Physical Sciences,
University of Wales, Aberystwyth

Modelling Foam Drainage

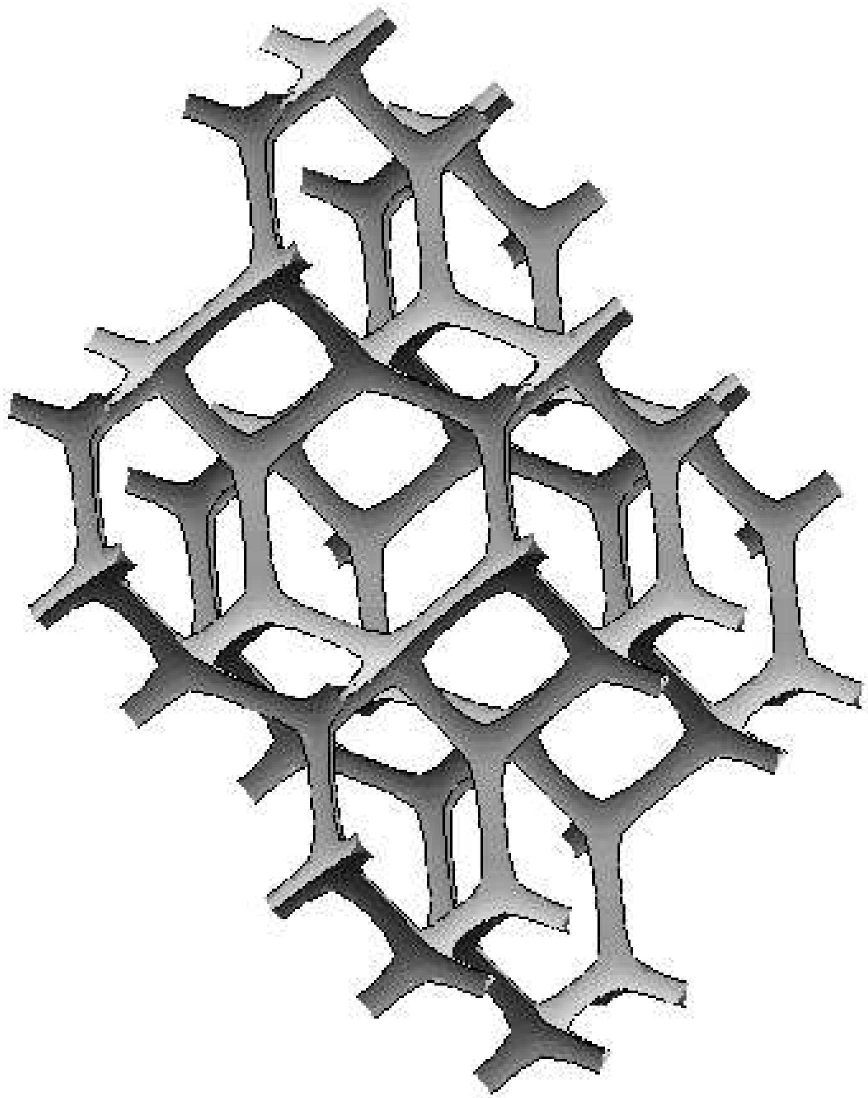
- Liquid flows through a static foam under the action of gravity.
- Viscosity and pressure differences affect the flow.
- Unlike in porous media, the foam can swell to accommodate more liquid.
- Need a model that describes the distribution of liquid in a foam over time.

Can we predict the results of experiments like this?



Capacitance measurements showing liquid fraction of a foam in free drainage. Time interval between scans was 20 seconds. [Weaire, Hutzler (1999) The Physics of Foams, Oxford.]

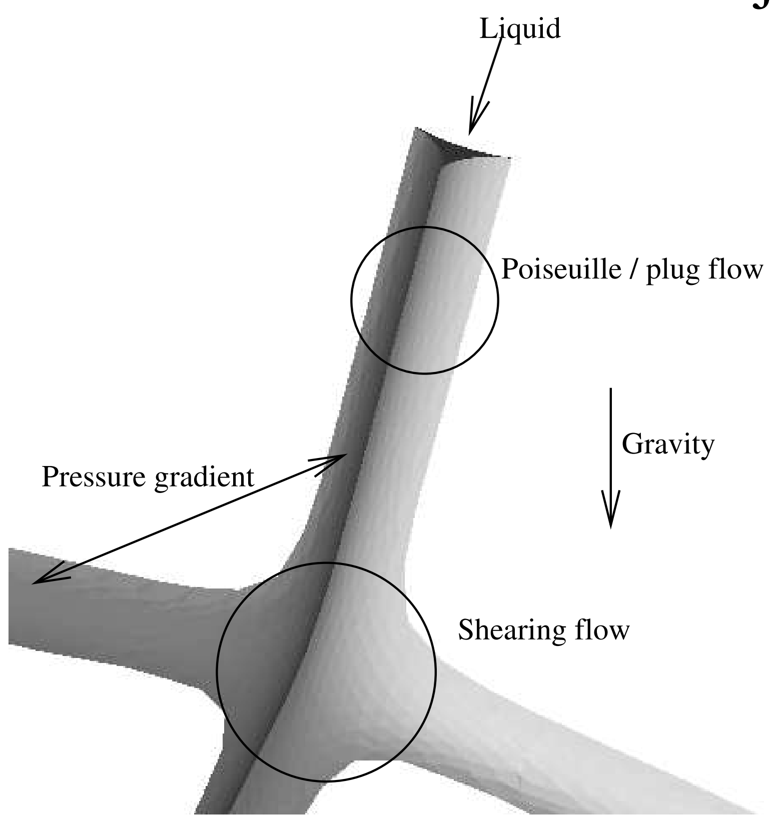
Liquid network



Assume low flow-rates and small capillary numbers, so that flow does not distort geometry of foam.

Two models

Two models: the choice of surfactant influences whether the dissipation occurs in the Plateau borders or the junctions.



Plateau border (channel-dominated) FDE

- Neglect the contribution of the liquid in the **films** and the **junctions**.
- Assume foam is **monodisperse** and **dry** (so Plateau borders are long and thin with approximately constant cross-sectional area).
- Assume **Poiseuille** (no-slip) flow in the Plateau borders (Stokes flow, no inertia).
- Define $A(\underline{x}, t)$ to be the Plateau border cross-section at position $\underline{x}$ and time t.
- A is a local measure of the foam's **liquid fraction**: $\Phi_l = cA/V_b^{2/3}$.
- The proportionality constant $c \approx 5.35$ depends weakly on structure.

Liquid conservation

- Consider a single PB through which liquid flows with flow-rate Q.

- Liquid is conserved, so

$$\frac{\partial A}{\partial t} + \nabla . Q = 0.$$

- To make the analysis more transparent, let's consider the flow in a vertical PB, $\underline{x} = (0, 0, z)$:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0.$$

Poiseuille flow in a PB

- What is the Stokes velocity profile in a PB with no-slip boundary conditions?

- If PB is long and thin, with constant A, then this is a 2D problem, $u = u(x, y)$.

- Solve Poisson's equation $\eta \nabla^2 u = G$ (pressure gradient), subject to $u = 0$ on the boundary of the curvilinear triangle.

- No exact solution, but
$$\langle u \rangle = \frac{1}{A} \int u dA = \frac{AG}{f \eta}$$
with a numerical prediction of the "drag coefficient" of $f \approx 49$ (cf $f = 8\pi$ for a circular pipe).

Force balance

- The flow-rate $Q = \langle u \rangle A$ is a balance between weight, viscous dissipation (drag) and pressure differences between parts of the PB with different A (capillary forces) .
- Recall that $A = C^2 r^2$, with $C^2 = 0.161$.
- For a volume element Adz of PB:
  - the weight of liquid is ρg;
  - the viscous force is $\eta f \langle u \rangle / A$;
  - the pressure difference, from the Laplace Young Law, $p_l - p_g = \frac{\gamma}{r}$, is

$$\frac{\partial p_l}{\partial z} = \frac{1}{2} C \gamma A^{-\frac{3}{2}} \frac{\partial A}{\partial z}$$

Flow-rate

- The force balance gives

$$\rho g = \frac{\eta f \langle u \rangle}{A} + \frac{1}{2} C \gamma A^{-\frac{3}{2}} \frac{\partial A}{\partial z}$$

so

$$Q = \langle u \rangle A = \frac{\rho g}{\eta f} A^2 - \frac{1}{2} \frac{C\gamma}{\eta f} \sqrt{A} \frac{\partial A}{\partial z}$$

- Not all Plateau borders are vertical, but randomly oriented. The flow-rate in a PB at angle θ is decreased by a factor of $\cos^2(\theta)$.

- Since $\langle \cos^2(\theta) \rangle = \frac{1}{3}$, there is an effective increase of the viscosity by a factor of 3.

Foam Drainage Equation

- The 1D channel-dominated foam drainage equation is therefore

$$3\eta f \frac{\partial A}{\partial t} + \frac{\partial}{\partial z}\left(\rho g A^2 - C\gamma \frac{\sqrt{A}}{2}\frac{\partial A}{\partial z}\right) = 0.$$

- Effective viscosity is $\eta^* = 3f\eta_l$; note how it affects only the time-scale.

- Boundary conditions are applied at top ($z = 0$) and bottom ($z = L$) of foam.

- This is a highly nonlinear equation: only a few analytic solutions have been found.

Dimensionless parameters

- Approximations valid up to about $\Phi_l = 5\%$; seems to work at higher Φ_l.

- For analysis and numerical solution, it is helpful to write the equation in dimensionless variables.

- e.g. put $\tau = t/t_0$, $\xi = z/z_0$ and $\alpha = A/z_0^2$ with $x_0 = \sqrt{\rho g/C\gamma}$ and $t_0 = 3f\eta/\sqrt{\rho g C\gamma}$; then

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial \alpha}{\partial \xi}\right) = 0.$$

Higher dimensions

- The equation generalizes naturally to 2D and 3D situations, since gravity acts only in the ξ direction:

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial \alpha^2}{\partial \xi} - \nabla . \left(\frac{\sqrt{A}}{2} \nabla A \right) = 0.$$

- We can also model changes in container width or coarsening by altering the number of Plateau borders $N(\underline{\xi}, \tau)$ in the foam:

$$\frac{\partial N \alpha}{\partial \tau} + \frac{\partial N \alpha^2}{\partial \xi} - \nabla . \left(N \frac{\sqrt{A}}{2} \nabla A \right) = 0.$$

Mostly numerical solutions required; exceptions include “Eiffel Tower”.

Node-dominated drainage

- Now consider a surfactant with low surface shear viscosity.
- Plug flow in PBs. Shearing flow in junctions.
- Integrate Stokes equation over the liquid in the foam:

$$\int_{V_{liq}} (\nabla p - \rho g) = \int_{V_n} \eta \nabla^2 \underline{u} dV + \int_{V_{pb}} \eta \nabla^2 \underline{u} dV.$$

Koehler et al (PRL, 1999)

- In this limit, PB contribution to viscous dissipation (last integral) is small.

Node-dominated drainage

- L is the length of a single Plateau border.
- Integrating over nodes (junctions), per unit volume of foam, gives
$$\rho g + C\gamma\nabla\frac{1}{\sqrt{A}} = \frac{\eta I}{2cL}\,\frac{\underline{u}}{\sqrt{A}}$$
where I is a dimensionless number independent of liquid fraction.
- In analogy with Darcy's Law for porous media, the permeability varies with $\sqrt{A}$. It is linear in the channel-dominated case.

Node-dominated drainage

- Projecting into 1D leads to a flow-rate

$$\frac{\eta I}{2cL} Q = \rho g A^{3/2} - \frac{1}{2} C\gamma \frac{\partial A}{\partial z}$$

which, combined with the continuity equation gives

$$\frac{\eta I}{2cL} \frac{\partial A}{\partial t} + \frac{\partial}{\partial z} \left(\rho g A^{3/2} - \frac{1}{2} C\gamma \frac{\partial A}{\partial z} \right) = 0.$$

- Differs from the channel-dominated equation by a factor of $\sqrt{A}$ in the flow-rate and the effective viscosity.

Dimensionless forms

- The final equations are very similar in dimensionless form:
- Channel-dominated:

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial \alpha}{\partial \xi}\right) = 0.$$

- Node-dominated:

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\alpha^{\frac{3}{2}} - \frac{1}{2}\frac{\partial \alpha}{\partial \xi}\right) = 0.$$

(Different non-dimensionalizations.)

Solutions of the drainage equations

- Steady, forced, free and pulsed drainage;
- Microgravity;

- Analytic solutions;
- Numerical solutions using finite difference (e.g. FORTRAN, C, . . .) or finite element (e.g. Femlab).

Steady drainage

- Constant liquid fraction is a solution of each FDE.
- It corresponds to a steady input of liquid into the top of a column of foam.
- Two other steady drainage solutions are found from each of

$$\alpha^2 - \frac{1}{2}\sqrt{\alpha}\frac{\partial \alpha}{\partial \xi} = \alpha_0^2$$

or

$$\alpha^{3/2} - \frac{1}{2}\frac{\partial \alpha}{\partial \xi} = \alpha_0^{3/2}$$

but hard to find physically realistic interpretations of them.

Equilibrium Profile

- Eventually a column of **freely draining** foam reaches **equilibrium**, when gravitational and pressure/capillary forces balance.

- Put flow-rate $Q = 0$ everywhere to get the **equilibrium profile**:

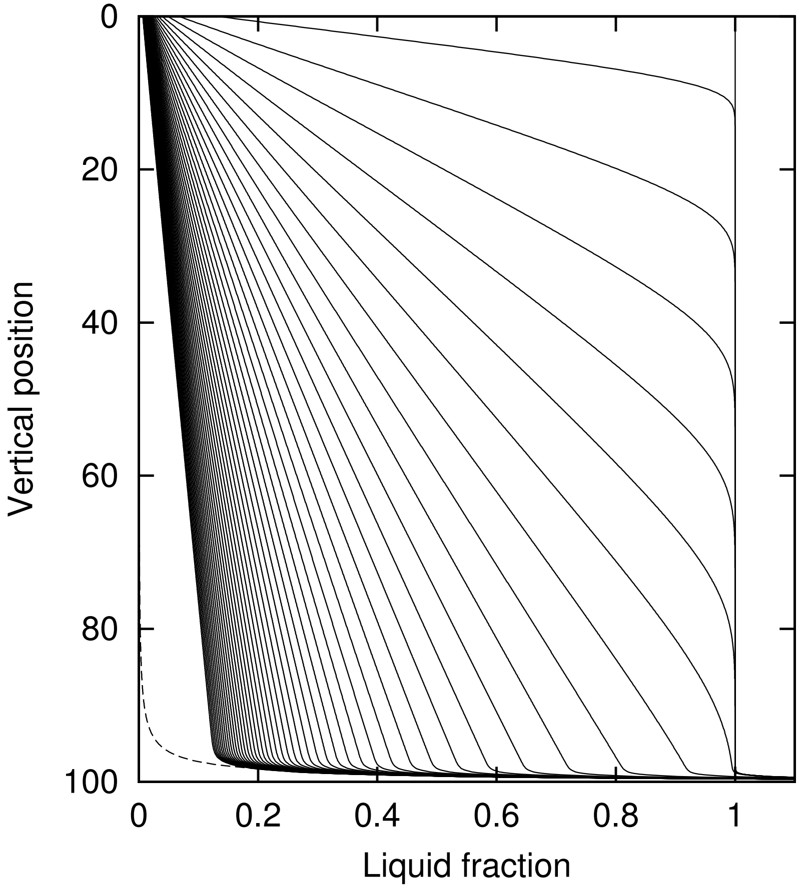
$$\alpha_{eq}(\xi) = \left(\frac{1}{\sqrt{\alpha_L}} + L - \xi \right)^{-2} .$$

- Liquid fraction **never** decreases completely to zero.

- Without coarsening/coalescence/collapse all liquid foams should attain this profile.

- How does it get there?

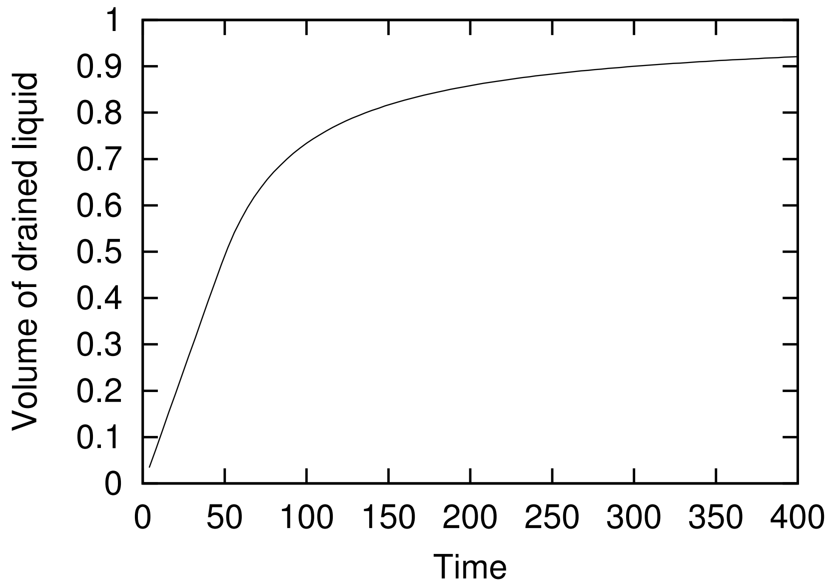
Free Drainage



- A foam of constant liquid fraction is made (e.g. using steady drainage) and left to drain under gravity.
- Use channel-dominated FDE (other case similar).
- The distribution of liquid varies in several well-defined regimes.

Drained liquid

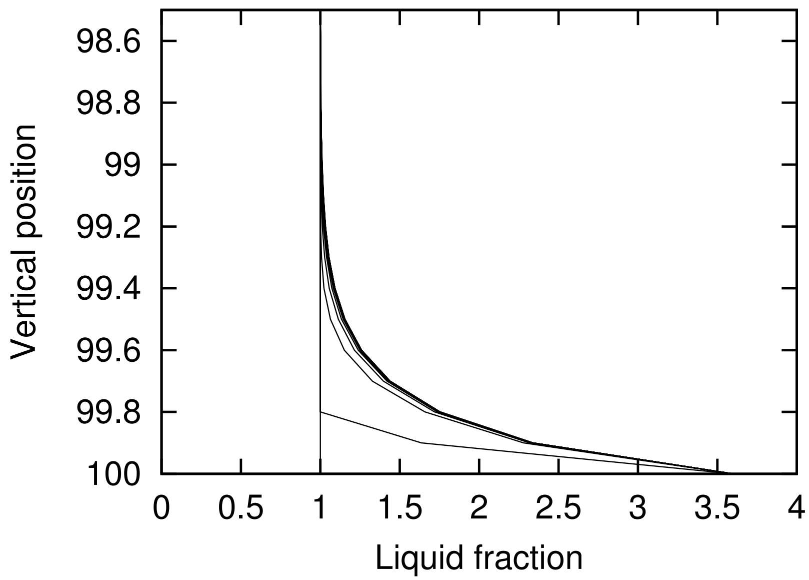
Volume of drained liquid $V(t)$ reflects liquid fraction profiles:

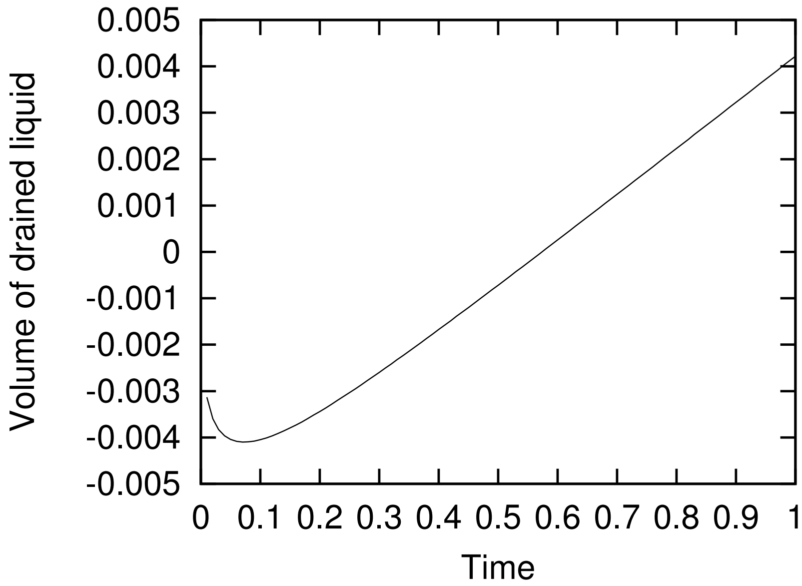


Capillary Rise (Regime 1)

- The profile at the base of the foam is established by capillary action.
- The liquid fraction has to increase from its initial value α_0 to the liquid fraction of a bubbly liquid α_L.
- Transition is almost instantaneous.
- Volume of liquid in foam **increases**.

Capillary Rise (Regime 1)





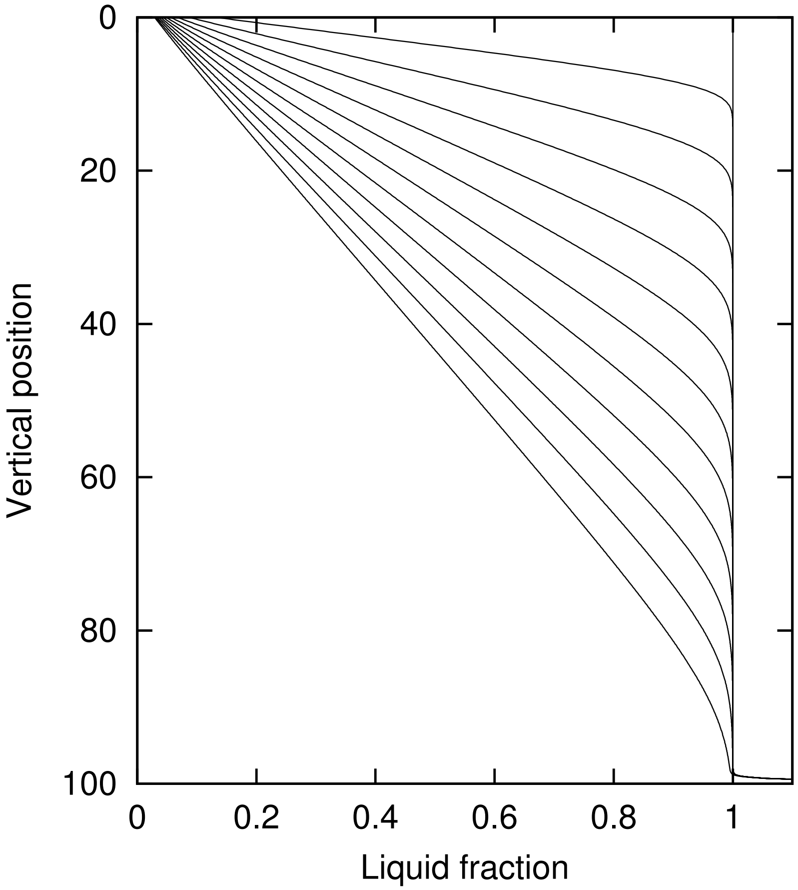
Kraynik's solution (Regime 2)

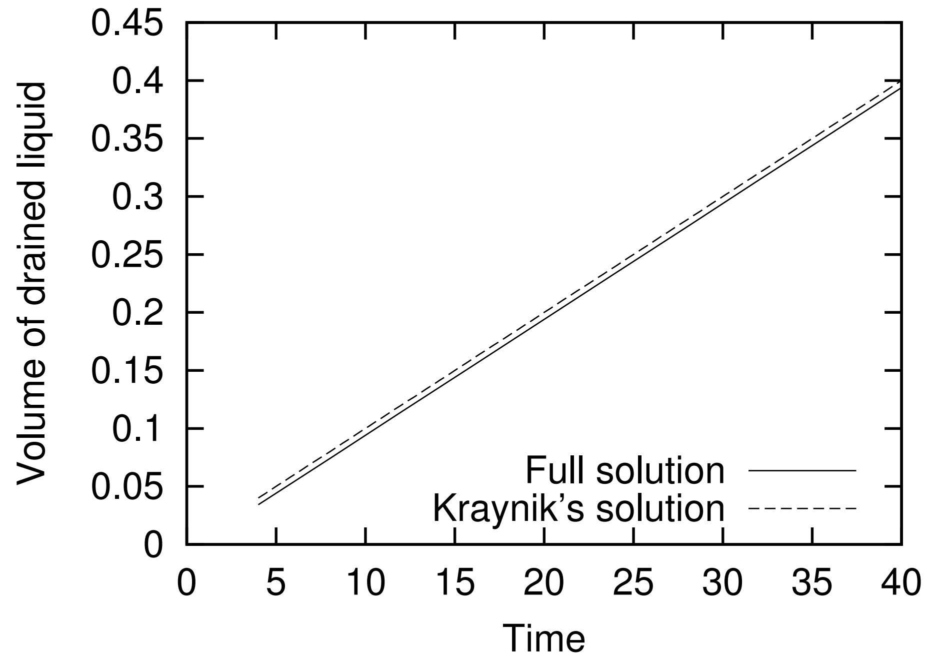
- Profiles of liquid fraction almost straight, so ignore capillary forces and solve
$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial \alpha^2}{\partial \xi} = 0$$
with boundary condition $\alpha(0, \tau) = 0$.
Can improve this: see Verbist, Weaire and Kraynik (1996).

- Kraynik's solution is
$$\alpha(\xi, \tau) = \begin{cases} \xi/(2\tau) & \xi \leq 2\alpha_0\tau \\ \alpha_0 & \xi \geq 2\alpha_0\tau \end{cases}$$
represents straight lines connecting the origin to the initial liquid fraction.

- Volume of drained liquid increases **linearly**: $V/V_0 = \frac{1}{L}\alpha_0\tau$.

Linear Increase in V (Regime 2)





Regime 3

- Profiles remain straight (away from bottom of foam) and evolve **independently** of initial liquid fraction.

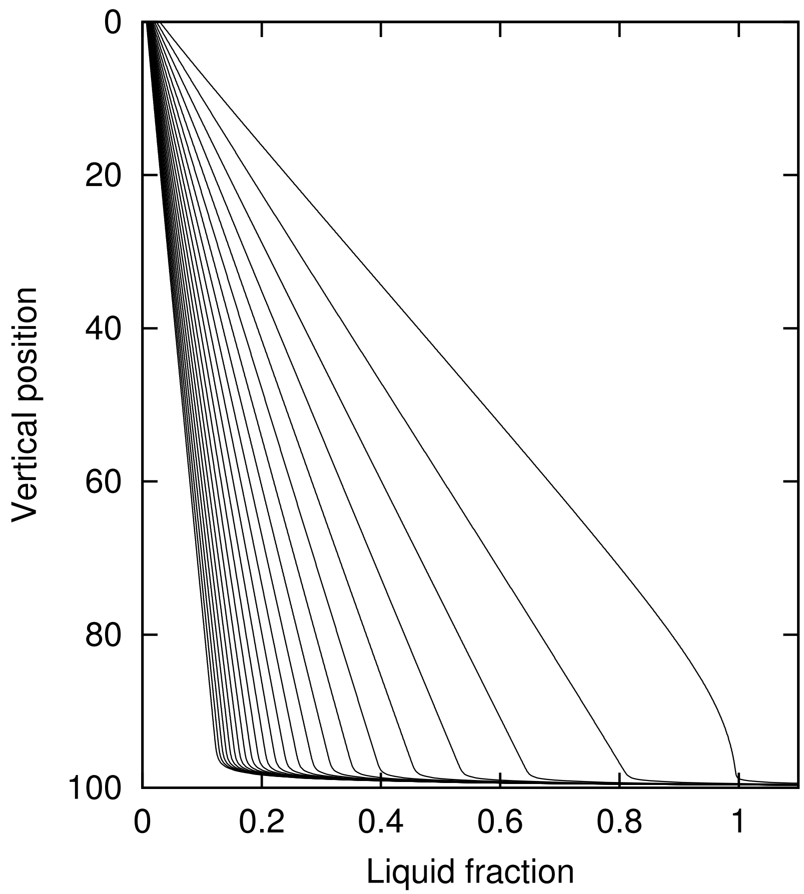
- Continue to use Kraynik's solution:

$$\alpha(\xi, \tau) = \frac{\xi}{2\tau}.$$

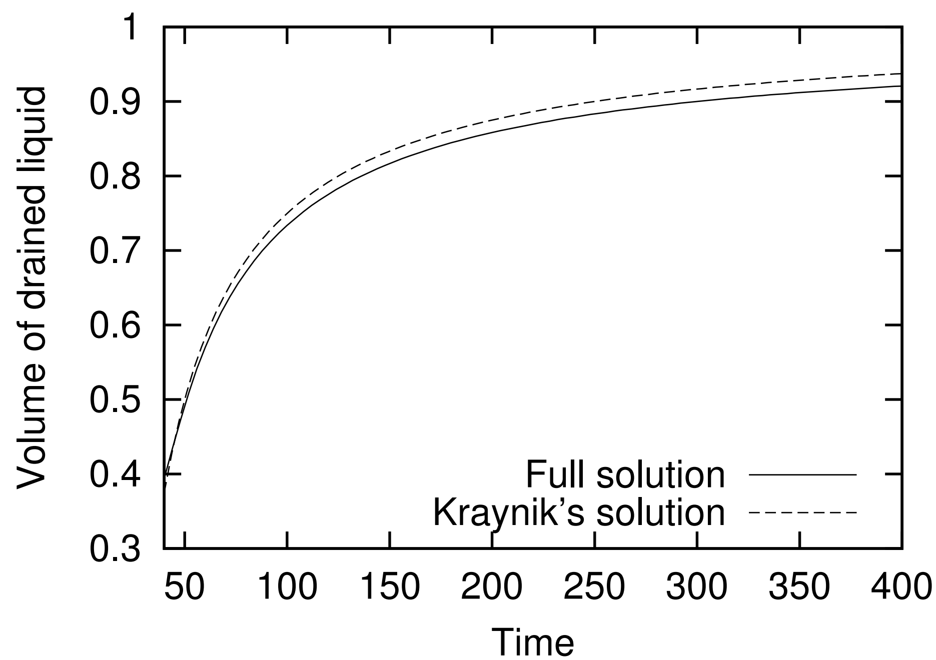
- Volume of drained liquid therefore

$$V/V_0 = 1 - \frac{L}{4\alpha_0}\frac{1}{\tau}.$$

Regime 3



Vertical position
Liquid fraction
0
20
40
60
80
100
0
0.2
0.4
0.6
0.8
1



Volume of drained liquid
Time
Full solution
Kraynik's solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
50
100
150
200
250
300
350
400

Approach to equilibrium I

- How does liquid fraction reach equilibrium profile?

- Evaluate the behaviour of a perturbation about equilibrium:

$$\alpha(\xi, \tau) = \alpha_{eq}(\xi) + \epsilon \alpha_p(\xi, \tau) + O(\epsilon^2)$$

  for $0 < \epsilon \ll 1$.

- Look for **separable** solutions

$$\alpha_p(\xi, \tau) = \bar{\alpha}_p(\xi) T(\tau)$$

  and substitute in FDE.

- Then $T(\tau) = \mathrm{e}^{-\mu\tau}$ for some $\mu > 0$.

- Approach to equilibrium is therefore **exponential**.

Approach to equilibrium II

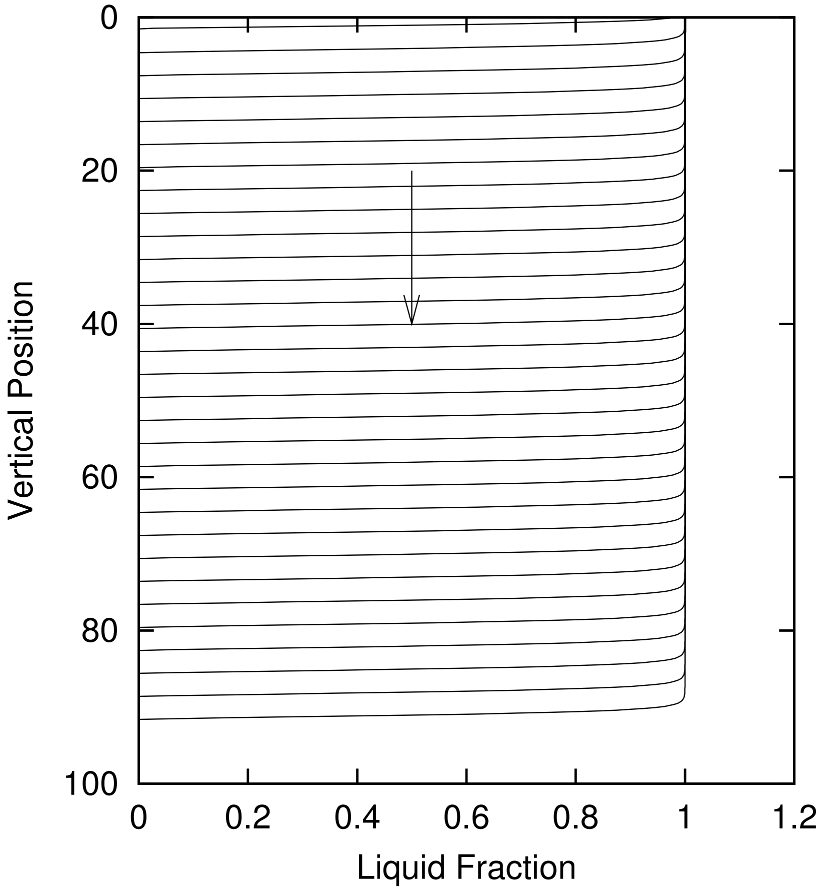
- We can construct the complete solution for $\alpha_p(\xi, \tau)$, which involves Bessel functions.

- The lowest eigenvalue is

$$\mu = \mu_0 = 1.22 \times 10^{-5}$$

giving a time-scale of the order of one hour.

Forced Drainage



- Liquid is added continuously to a dry foam.
- It is observed to descend with a well-defined **front** at **constant speed** v.

- FDEs have a **travelling wave** solution.

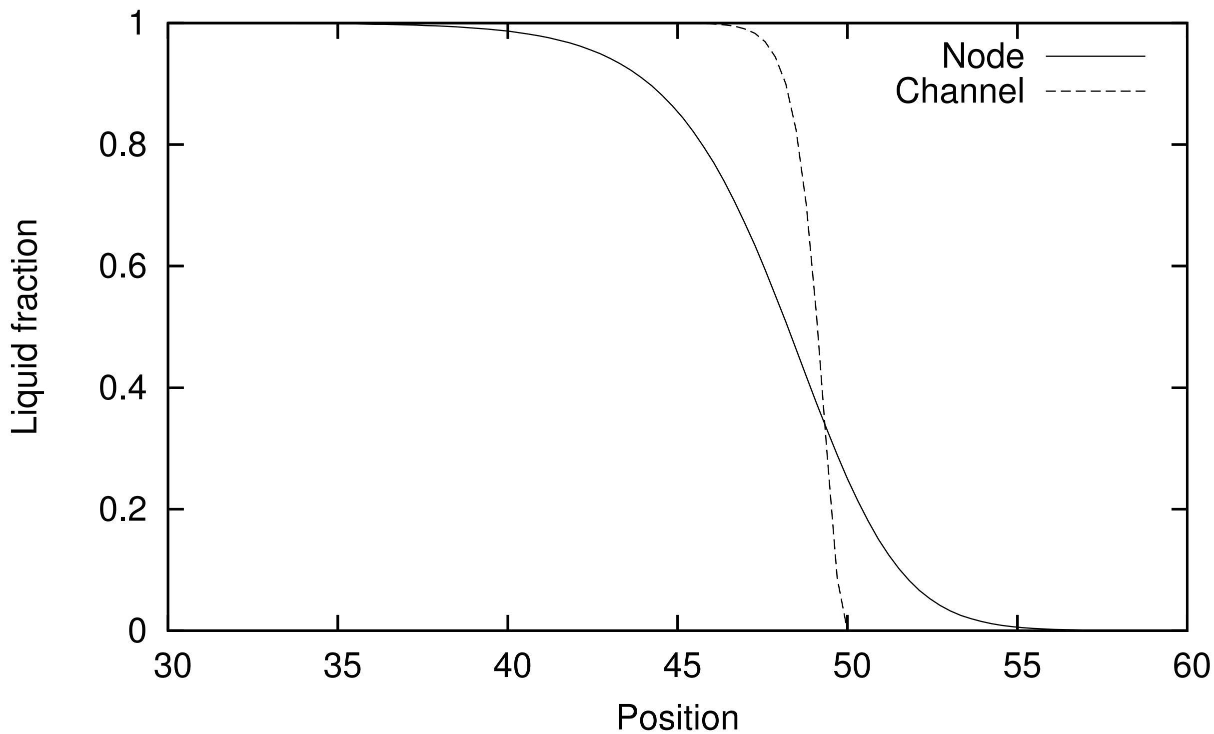
- Channel-dominated:

$$\alpha(\xi,\tau) = \begin{cases} v\,\mathsf{tanh}^2(\sqrt{v}[\xi - v\tau]) & \xi \le v\tau \\ 0 & \xi \ge v\tau \end{cases}$$

- Node-dominated:

$$\alpha(\xi,\tau) = \left[\frac{v}{1 + \mathsf{exp}(\frac{1}{2}v[\xi - v\tau])}\right]^2$$

Solitary wave

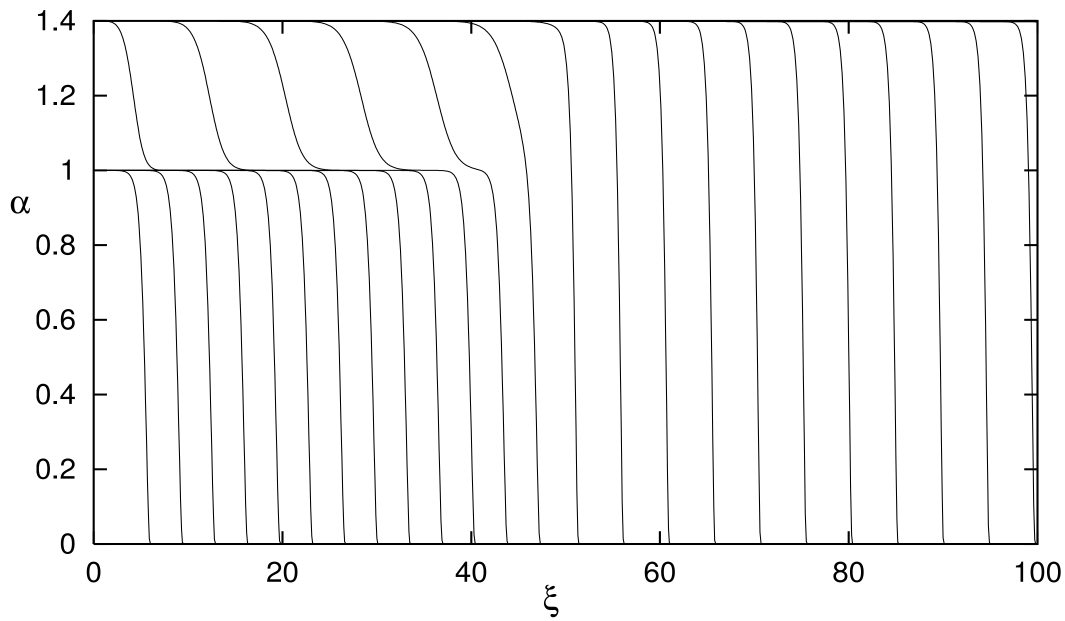


- Qualitative agreement excellent; quantitative agreement to within a factor of 2.
- Note that wave may actually run on something closer to equilibrium background.

Important differences

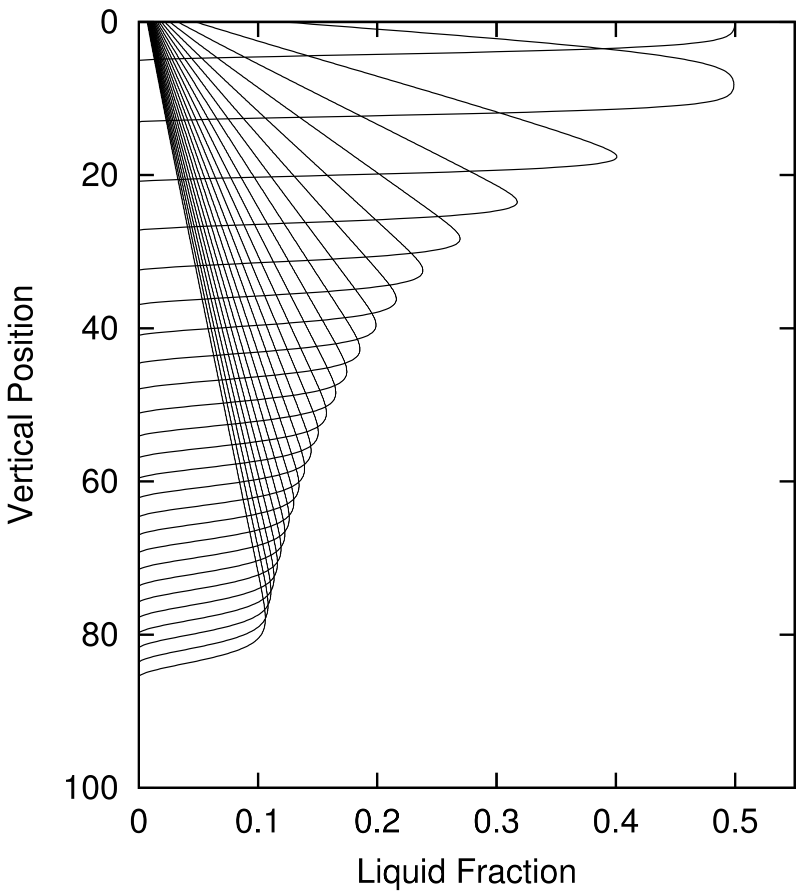
- Channel-dominated: $v(=\alpha) \sim Q^{1/2}$; $w_f \sim v^{-1/2}$.
- Node-dominated: $v \sim Q^{1/3}$; $w_f \sim v^{-1}$.
- Latter is exponential, so front not sharp.
- Identify Q with superficial velocity to eliminate container effects.

Related solutions include a wave on a uniformly wet foam (double wave):



The velocities add.

Pulse



A small volume V of liquid is added to the top of a foam and left to evolve.

Pulse II

- By ignoring the capillary (diffusive) terms we can use **Kraynik's solution** again:

$$\alpha = \begin{cases} \dfrac{\xi}{2\tau} & \xi \le 2\sqrt{V\tau} \\ 0 & \xi > 2\sqrt{V\tau} \end{cases}$$

i.e. straight lines radiating from the origin (N-waves).

- In the node-dominated case this is

$$\alpha = \begin{cases} \dfrac{4\xi^2}{9\tau^2} & \xi \le 3(\frac{1}{4}V\tau^2)^{1/3} \\ 0 & \xi > 3(\frac{1}{4}V\tau^2)^{1/3} \end{cases}$$

so that the profile is quadratic.

With no gravitational force, we can study wet foams experimentally.

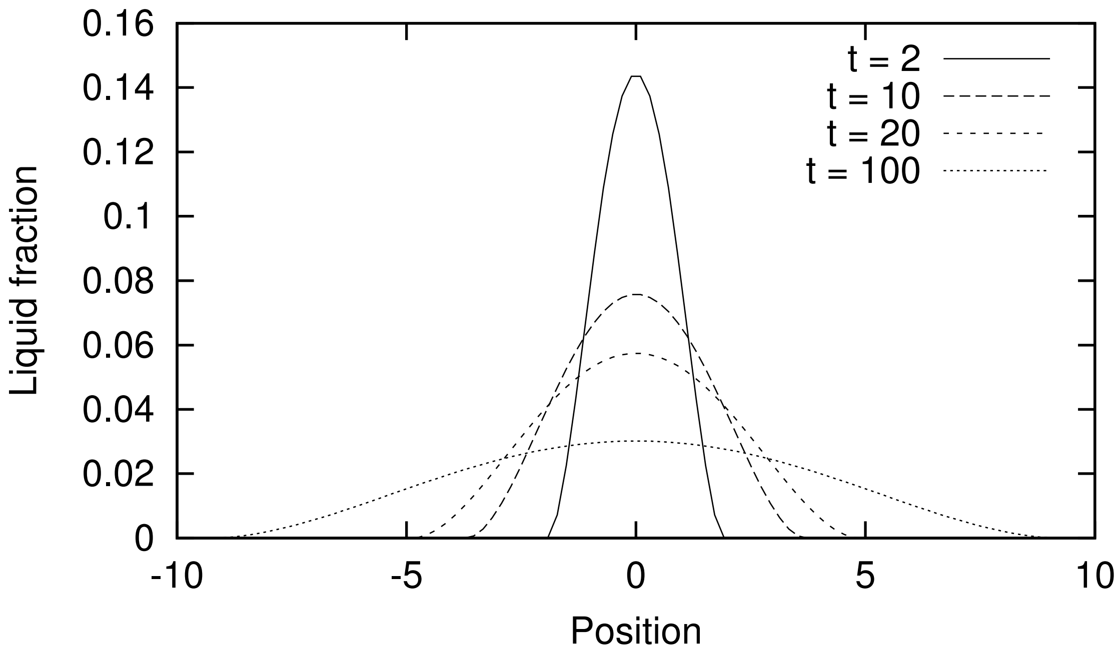
The channel-dominated FDE becomes

$$\frac{\partial \alpha}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{\sqrt{\alpha}}{2}\frac{\partial \alpha}{\partial \xi}\right)$$

i.e. a nonlinear diffusion equation. Node-dominated is linear.

Analogies of free, forced and pulsed drainage, with relevant scaling solutions.

A spreading pulse in zero gravity



A spreading pulse in zero gravity

A small volume λ_0 of liquid is introduced into a dry foam and left to spread. Conservation of liquid requires that the solution scales according to

$$\alpha(\xi, \tau) = \tau^{-2/5} y(s), \quad s = \frac{\xi}{\tau^{2/5}}.$$

Then y satisfies

$$-(sy)' = \frac{5}{6}\left(y^{3/2}\right)''$$

with solution

$$\alpha(\xi, \tau) = \frac{\left(C^2 \tau^{4/5} - \xi^2\right)^2}{25\tau^2}$$

where $C = (375\lambda_0/16)^{1/5}$.

[Koehler, Hilgenfeldt & Stone (1999) in *Foams and Films* MIT Verlag.]

Summary

- There are now several drainage equations that can be applied in different situations, predicting behaviour in good agreement with experiments on dry foams.
- Recent theory can account for **bubble motion**, non-uniform bubble **distribution**, changes in **container shape** and **microgravity**.
- There are still some unexplained effects, particularly instabilities of the foam structure.