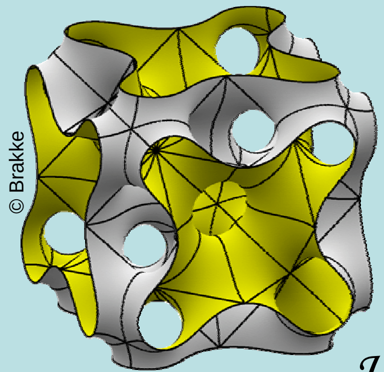
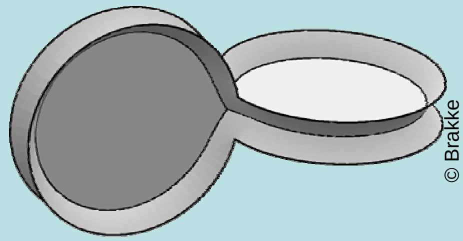
Minimal Surfaces

(or at least a flavour thereof)





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Soap Films

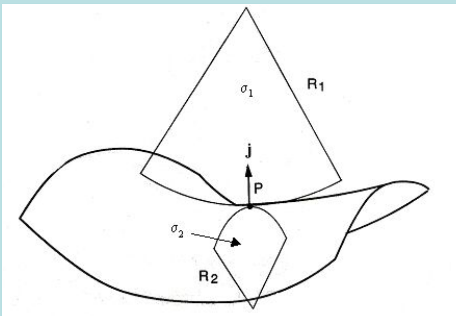
The force of surface tension in a single soap film causes it to reduce its area.

What shapes can a film take, and how can we describe them?

How can soap films join?

Radii of Curvature

A small element of soap film can always be locally approximated to have two principal radii of curvature, or normal curvatures, R_1 and R_2.



Young and Laplace (c1800) showed that these are related to the pressure difference across the film and its surface tension:

$$p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Curvature

Mean curvature:

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

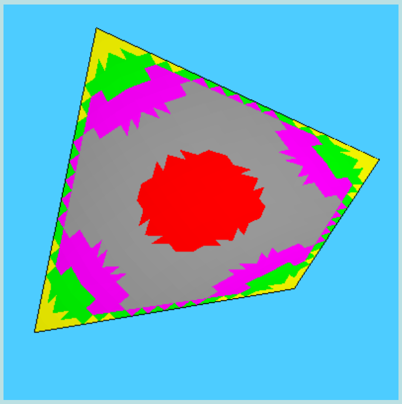
(Note factor of 2)

Gaussian curvature:

$$K = \frac{1}{R_1 R_2}$$

(intrinsic curvature)

Curvature



H=0 does **not** imply *K=0*

(colours denote K^2)

$$H^2 \geq K$$

K=0 on a flat surface

$R_1 = R_2$ at umbilic points,

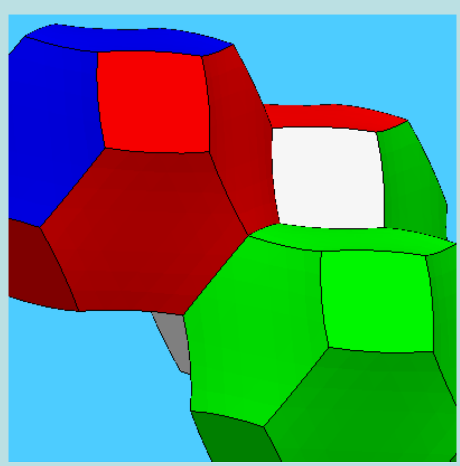
where $H^2 = K$.

Minimal Surfaces

A minimal surface is one for which *H=0*.

The pressure difference across it is therefore zero.

Pressure difference non-zero gives a constant mean curvature (cmc) surface.

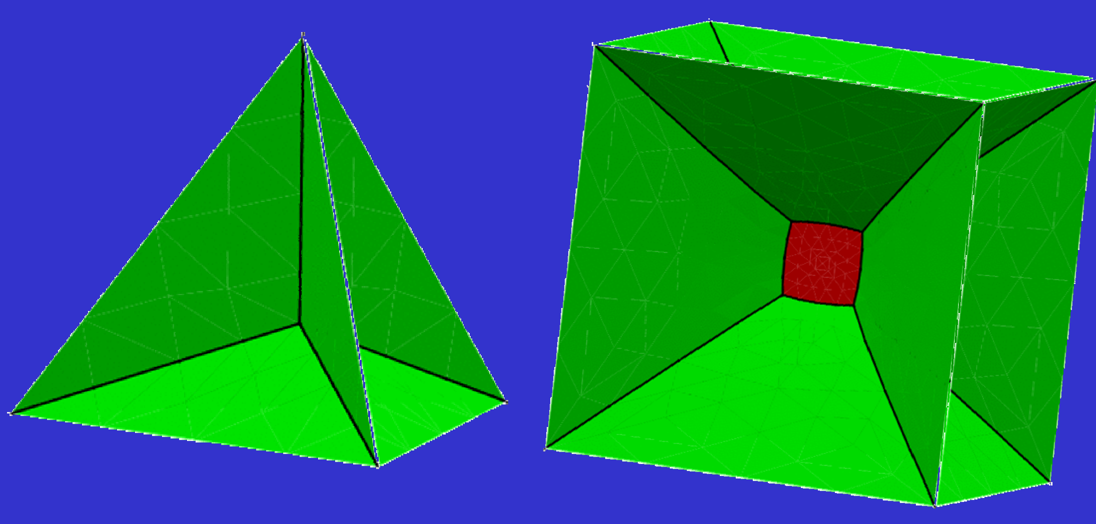


Plateau's Rules

Taylor (1976) showed, using geometric measure theory, that Plateau's rules for a dry foam are a consequence of the fact that soap films are minimal surfaces.

The singular set consists of Hölder continuously differentiable curves (PBs) at which three sheets (soap films) of the surface meet Hölder continuously at equal angles; four curves and six sheets of the surface meet Hölder continuously at each singular point at equal angles.

Tetrahedral and Cubic Frames



For each film, calculate shape that gives surface of zero mean curvature.

Steiner problem

Soap films in 2D solve the Steiner problem:

Given *n* cities on a plain (ignoring rivers etc.), what is the arrangement of roads joining them with minimum length?

Geometrical proofs long-winded and complex; experimental demonstration straightforward

Bubbles

A single bubble is spherical (isoperimetric inequality)

We can parametrize the surface by

$\boldsymbol{x}(u,v) = (R\cos(u)\cos(v), R\sin(u)\cos(v), R\sin(v))$.

Radii of curvature are R and R, so $H=1/R$ and $K=1/R^2$.

More generally:

A closed surface which minimizes surface area subject to a fixed enclosed volume must have cmc.

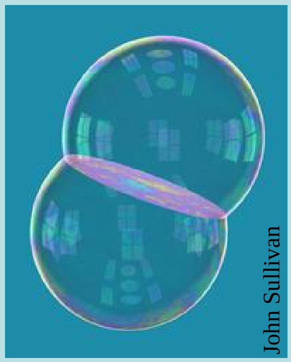
Therefore all bubbles must have cmc surfaces.

Bubbles

The double bubble *is* the minimum surface area partition of two volumes in $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, …

(See Morgan's *Geometric Measure Theory*)

Proof of triple bubble in 2D, and no more?



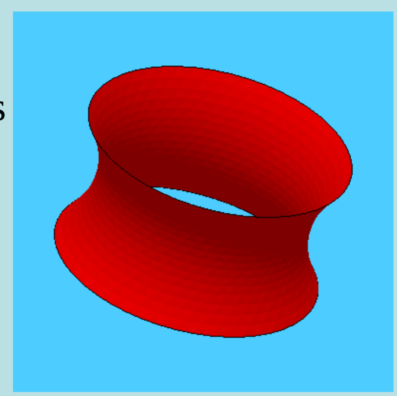
How many bubbles must there be in a 3D cluster before it contains a soap film that is not a spherical cap?

Circular rings

What is the shape of the minimal surface joining two circular rings?

The ***catenoid*** is the surface of revolution generated by $y = cosh(x)$. It is parameterized by $\mathbf{x}(u,v) = (cosh(u)cos(v), cosh(u)sin(v), u)$.

Finding tangents and normals to surface enables us to show that $H=0$.



Minimal Surface Equation

Consider a surface M given by $z = f(x,y)$ and parametrize it by $\mathbf{x}(u,v) = (u,v,f(u,v))$.

Then $$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{\left(1 + f_u^2 + f_v^2\right)^2}$$

and $$H = \frac{\left(1 + f_v^2\right) f_{uu} + \left(1 + f_u^2\right) f_{vv} - 2 f_u f_v f_{uv}}{2\left(1 + f_u^2 + f_v^2\right)^{3/2}}$$

Minimal Surface Equation

Therefore M is a minimal surface if and only if

$$\left(1 + f_v^2\right) f_{uu} - 2 f_u f_v f_{uv} + \left(1 + f_u^2\right) f_{vv} = 0$$

Hard to solve, in general.

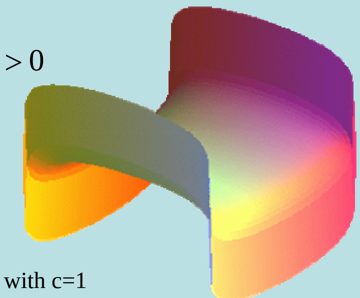
Scherk's first surface

Only third minimal surface discovered
(1835, after Catenoid and Helicoid)

Separable solution of minimal surface equation:

$$f(x, y) = c \ln\left(\frac{\cos(x/c)}{\cos(y/c)}\right)$$

defined on rectangles with $\frac{\cos(x/c)}{\cos(y/c)} > 0$



e.g. with c=1

Plateau's problem

Does any shape of boundary enclose a soap film?

or

Given a closed curve C find a minimal surface M with C as boundary.

Solved early in 20th century for "nice" (Jordan) curves.

NB: area-minimizing implies minimal
(counter example to converse is Enneper's surface)

Weierstrass-Enneper representation

Best approach to finding minimal surfaces.

Given "nice" complex functions *f* and *g* on a domain *D,* a minimal surface may be defined by the parameterization

$$\mathbf{x}(z) = \left(x^1(z), x^2(z), x^3(z)\right), \quad z = u + iv$$

where
$$x^1(z) = \Re\int f(1-g^2)\ dz$$
$$x^2(z) = \Re\int if(1+g^2)\ dz$$
$$x^3(z) = \Re\int 2fg\ dz$$

Weierstrass-Enneper representation

Best approach to finding minimal surfaces.

Given an analytic function *F,* a minimal surface may be defined by

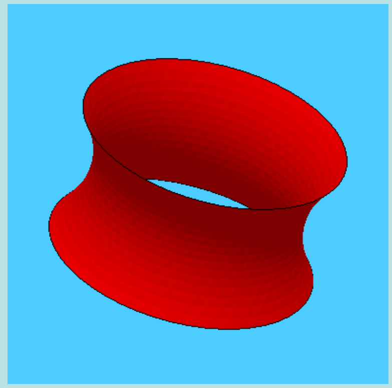
$$x^1(z) = \Re\int (1-\tau^2)F(\tau)\ d\tau$$
$$x^2(z) = \Re\int i(1+\tau^2)F(\tau)\ d\tau$$
$$x^3(z) = \Re\int 2\tau F(\tau)\ d\tau$$

Choose a function and, if you can do the integrals, you have a minimal surface

Example

Choose $F(\tau) = 1/2\tau^2$.

$$x^1 = \Re\int (1-\tau^2)\frac{1}{2\tau^2}d\tau$$
$$= \Re\int \frac{1}{2\tau^2} - \frac{1}{2}d\tau$$
$$= -\Re\left[\frac{1}{2\tau} + \frac{\tau}{2}\right]$$
$$= -\Re\left[\frac{e^{-z}}{2} + \frac{e^{z}}{2}\right]$$
$$= -\Re[\cosh(z)]$$
$$= -\cosh(u)\cos(v)$$



Continuing in this way shows that $\mathbf{x}(u,v) = (-\cosh(u)\cos(v), -\cosh(u)\sin(v), u)$.

This is the catenoid

Gauss Map

A mapping G of a surface M to the unit sphere $S^2 \subset R^3$

$G(p)$ is the unit normal to M at p

(*cf* the sphere parameterization earlier).

If M is a minimal surface then G is conformal.

If G is conformal then M is either minimal or part of a sphere.

Schwarz reflection principles

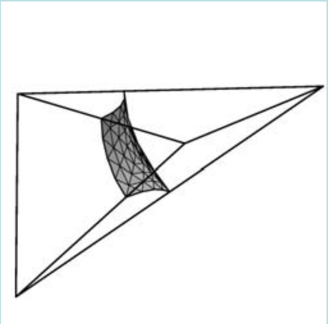
A minimal surface is symmetric about any straight line contained in the surface

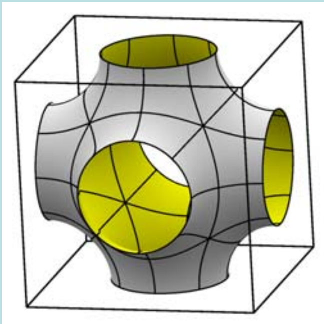
A minimal surface is symmetric about any plane which intersects the surface orthogonally

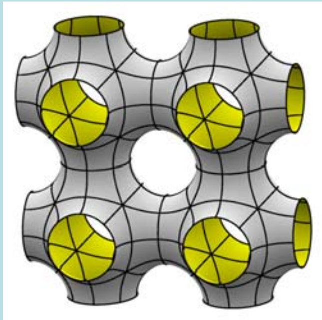
Triply Periodic Minimal Surfaces

Schwarz reflection symmetries can be used to generate minimal surfaces that repeat themselves in 3D.

For example, Schwarz' P-surface:



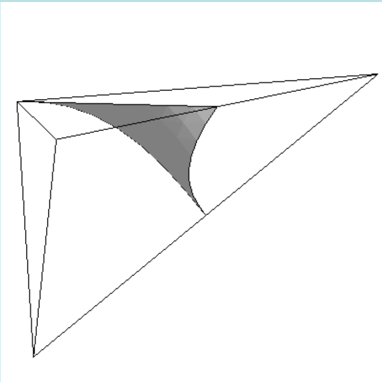


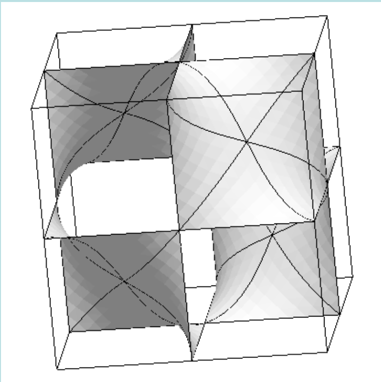


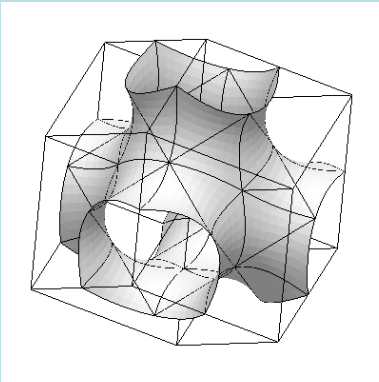
See Evolver webpage

Triply Periodic Minimal Surfaces

Schwarz' D-surface:





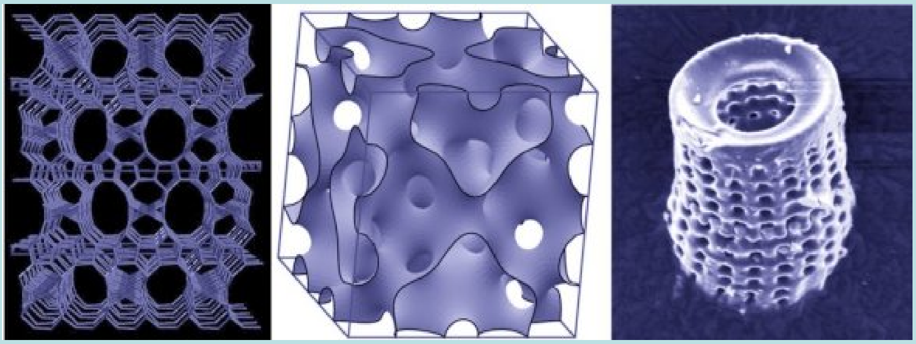


There are many others!

See Evolver webpage

Microporous Materials

Microporous titanosilicate ETS-10, mesoporous SBA-1 and macroporous diatomaceous earth



Source: University of Manchester Centre for Microporous Materials

In Zeolites (microporous aluminophosphates), used in washing powders, the atoms lie on minimal surfaces (surfaces of constant potential).

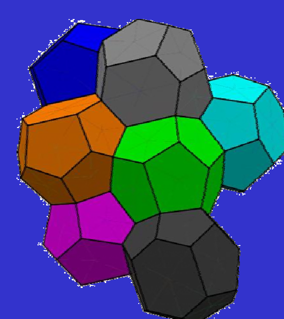
Numerical approaches

Weierstrass-Enneper representation
e.g. in Maple or Mathematica

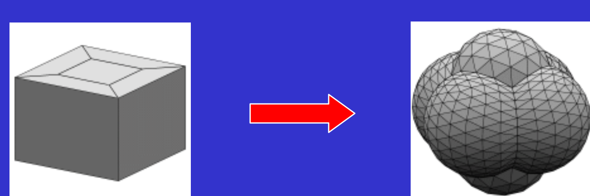
or

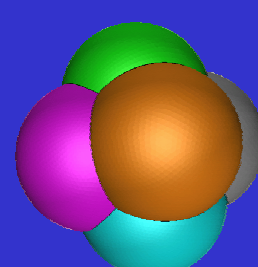
Minimize energy (area) directly,
e.g. Surface Evolver (accuracy) or
Potts Model (speed, statistics)

Ken Brakke's Surface Evolver



"The Surface Evolver is software expressly designed for the modeling of soap bubbles, foams, and other liquid surfaces shaped by minimizing energy subject to various constraints …"





http://www.susqu.edu/brakke/evolver/

Simulations of static structure

Surface Evolver works by representing a surface as a collection of triangles.

Text file contains vertex positions, edge connectivity and face data.

Triangulation is performed automatically, and can be refined.

Iterate towards equilibrium by perturbing vertices and retaining perturbations that reduce energy.

```
// quad.fe
// simple skew
// quadrilateral

vertices
1   0  0  1   fixed
2   2  0  0   fixed
3   2  2  1   fixed
4   0  2  0   fixed

edges
1   1  2 fixed
2   2  3 fixed
3   3  4 fixed
4   4  1 fixed

faces
1   1 2 3 4
```

Simulations of static structure

To represent a bubble, need to add a volume constraint V_0. This is represented as an addition to the energy:

$$E = \gamma A + p(V - V_0).$$

The Lagrange multiplier p is the bubble pressure.

Extensive command language allows surfaces to be manipulated and data to be recorded.

See tutorials!

```
// cube.fe
// Single bubble
vertices
1  0.0 0.0 0.0
2  1.0 0.0 0.0
3  1.0 1.0 0.0
4  0.0 1.0 0.0
5  0.0 0.0 1.0
6  1.0 0.0 1.0
7  1.0 1.0 1.0
8  0.0 1.0 1.0

edges
1   1 2
2   2 3
3   3 4
4   4 1
5   5 6
6   6 7
7   7 8
8   8 5
9   1 5
10  2 6
11  3 7
12  4 8

faces
1   1 10 -5  -9
2   2 11 -6 -10
3   3 12 -7 -11
4   4  9 -8 -12
5   5  6  7   8
6  -4 -3 -2  -1

bodies
1   1 2 3 4 5 6  volume 1
```

Initial Conditions

• Often inconvenient to construct an initial data file by hand.

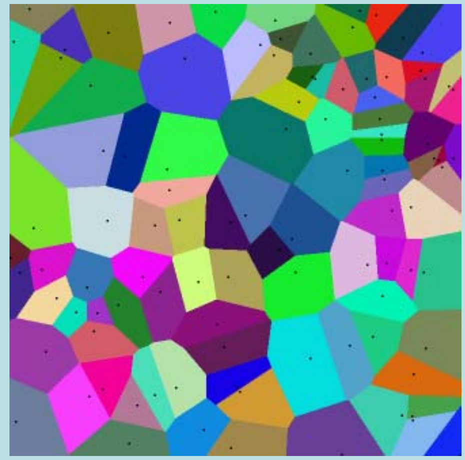
• Most popular algorithm is a Voronoi construction:

- scatter a number of points in space;

- each point S_i generates a Voronoi cell: those points that are closer to S_i than any other S_j.

• Use software such as Sullivan's VCS (http://torus.math.uiuc.edu/jms/software/).

• Then relax to equilibrium in Potts Model, Surface Evolver, etc.



Other possibilities - lattice? Annealing?

Summary

• Soap films are surfaces of constant mean curvature.

• Minimal surfaces are those that have zero mean curvature.

• Geometric measure theory provides the mathematical tools to describe the structure of foam rigorously, and in the context of other structures of co-dimension one.