MA22S3 Tutorial Sheet 9: Solutions

7–8 December 2016

1. Use the method of variation of parameters to find the solution to the following initial value problem.

$$y'' - 2y' + y = 2xe^x, \qquad y(0) = -1, \quad y'(0) = 0.$$

Solution: The solutions to the complementary equation $y'' - 2y' + y = 0$ can be found from the trial function $y = e^{\lambda x}$, giving the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ with a double root at $\lambda = 1$. Our two complementary solutions are therefore

$$y_1 = e^x, \qquad y_2 = xe^x.$$

To solve the nonhomogeneous differential equation using variation of parameters, we write a particular solution in the form

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= u_1 e^x + u_2 x e^x. \end{aligned}$$

The first derivative of this expression is

$$y_p' = u_1 e^x + u_2(x+1)e^x + u_1' e^x + u_2' x e^x$$

At this point, we impose the additional constraint that the terms with u_1' and u_2' sum to zero:

$$\begin{aligned} u_1' e^x + u_2' x e^x &= 0 \\ u_1' + u_2' x &= 0 \end{aligned} \tag{1}$$

Then the second derivative of y_p is

$$y_p'' = u_1 e^x + u_2(x+2)e^x + u_1' e^x + u_2'(x+1)e^x$$

Substituting these derivatives into the differential equation, we find that the terms proportional to u_1 and u_2 cancel out, as expected, leaving us with the equation

$$\begin{aligned} u_1' e^x + u_2'(x+1)e^x &= 2xe^x \\ u_1' + (x+1)u_2' &= 2x \end{aligned} \tag{2}$$

Therefore

$$u_1' = 2x - (x+1)u_2'.$$

We can solve the linear system of equations (1) and (2) for u_1' and u_2', with the result

$$\begin{aligned} u_1' &= -2x^2 \\ u_2' &= 2x \end{aligned}$$

Alternatively, we could have written these solutions for u_1' and u_2' more directly, using the formula given in terms of the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$.

Integrating these solutions while disregarding the constants of integration, we find

$$u_1 = -\frac{2}{3}x^3, \qquad u_2 = x^2.$$

Therefore, a particular solution to the differential equation is given by

$$\begin{aligned} y_p &= -\frac{2}{3}x^3e^x + x^3e^x \\ &= \frac{1}{3}x^3e^x \end{aligned}$$

The general solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 + y_p \\ &= c_1e^x + c_2xe^x + \frac{1}{3}x^3e^x\,. \end{aligned}$$

Now let us fix the constants with the initial values.

$$\begin{aligned} y'(x) &= c_1e^x + c_2(x+1)e^x + \left(\frac{1}{3}x^3 + x^2\right)e^x \\ -1 &= y(0) = c_1 \\ 0 &= y'(0) = c_1 + c_2 \end{aligned}$$

Therefore $c_1 = -1$ and $c_2 = 1$, so the solution is

$$y = -e^x + xe^x + \frac{1}{3}x^3e^x\,.$$

2. Find the first four terms in the series expansion about $x = 0$ of the solution to the following initial value problem.

$$y' + 2xy = e^x, \qquad y(0) = 1.$$

Solution: Write the series expansion for the solution as

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Differentiate to get

$$y' = \sum_{m=0}^{\infty} m a_m x^{m-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Then

$$\begin{aligned} 2xy &= \sum_{m=0}^{\infty} 2a_m x^{m+1} \\ &= \sum_{m=2}^{\infty} 2a_{m-2}x^{m-1} \\ &= 2a_0x + 2a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots \end{aligned}$$

We also need the (Taylor) series expansion for e^x about $x = 0$, which is

$$\begin{aligned} e^x &= \sum_{m=0} \frac{1}{m!}x^m = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ &= 1 + \sum_{m=1} \frac{1}{(m-1)!}x^{m-1} \end{aligned}$$

Equating coefficients of like powers of x in the differential equation, we find

$$\begin{aligned}
[x^0]: &\qquad a_1 = 1\\
[x^1]: &\qquad 2a_2 + 2a_0 = 1\\
[x^2]: &\qquad 3a_3 + 2a_1 = \frac{1}{2}
\end{aligned}$$

Given that $a_0 = y(0) = 1$, and value $a_1 = 1$, we can compute $a_2 = -1/2$ and $a_3 = -1/2$. Therefore the series expansion around $x = 0$ starts with

$$y = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots$$

It is possible to write a formula for the recursive relation producing the coefficients of the power series. To do this, we write

$$\begin{aligned}
y &= \sum_{m=0}^{\infty} a_m x^m\\
y' &= \sum_{m=0}^{\infty} m a_m x^{m-1}\\
2xy &= \sum_{m=0}^{\infty} 2a_m x^{m+1} = \sum_{m=2}^{\infty} 2a_{m-2} x^{m-1}\\
e^x &= \sum_{m=0} \frac{1}{m!} x^m = 1 + \sum_{m=1} \frac{1}{(m-1)!} x^{m-1}
\end{aligned}$$

The summation indices have been shifted so that we can write the series solution by equating powers of x^{m-1} on both sides, starting from $m \geq 2$. The first two terms need to be treated separately. We find

$$\begin{aligned}
[x^0]: &\qquad 1 = 1\\
[x^1]: &\qquad a_1 = 1\\
[x^{m-1}],\ m \geq 3: &\qquad m a_m + 2a_{m-2} = \frac{1}{(m-1)!}.
\end{aligned}$$

The recursive relation is

$$\begin{cases}
a_0 \text{ is undetermined,}\\
a_1 = 1.\\
a_m = \frac{1}{m}\left(\frac{1}{(m-1)!} - 2a_{m-2}\right) \quad \text{for } m \geq 2.
\end{cases}$$