MA22S3 Tutorial Sheet 8: Solutions

30 November – 1 December 2016

1. Find the general solution to
$$h''(x) + \frac{2}{x}h'(x) + \frac{3}{x^2}h(x) = 0.$$

Solution:

This equation is of the Euler-Cauchy type. It can be rewritten as $x^2h''(x) + 2xh'(x) + 3h(x) = 0.$. Use a trial solution $h = x^m$. The auxiliary equation is $m(m-1) + 2m + 3 = 0$, which has distinct real roots at $(-1 \pm i\sqrt{11})/2$. So the general solution can be written as
$$h(x) = c_1\, x^{-\frac{1}{2}+\frac{\sqrt{11}}{2}i} + c_2\, x^{-\frac{1}{2}-\frac{\sqrt{11}}{2}i}$$
or alternatively, in terms of real functions,
$$h(x) = c_3\frac{\cos\left(\frac{\sqrt{11}}{2}\log x\right)}{\sqrt{x}} + c_4\frac{\sin\left(\frac{\sqrt{11}}{2}\log x\right)}{\sqrt{x}}$$

2. Find the general solution to
$$x^2g''(x) - xg'(x) + g(x) = x^6.$$

Solution:

First step: find two linearly independent complementary solutions to the homogeneous equation $x^2g''(x) - xg'(x) + g(x) = 0$. Since this is an Euler-Cauchy equation, use the trial function $g = x^m$. The characteristic equation is $m(m-1) - m + 1 = 0$, which has a double root at $m = 1$. So the two independent complementary solutions, are
$$g_1 = x, \qquad g_2 = x\log x.$$
Second step: find a particular solution to the nonhomogeneous equation. The trial function is the same type of power as appears in the nonhomogeneous part, with an undetermined coefficient:
$$g_p = Ax^6.$$
The first and second derivatives of this trial function are
$$\begin{aligned} g_p' &= 6Ax^5 \\ g_p'' &= 30Ax^4. \end{aligned}$$
We substitute these functions into the differential equation:
$$30Ax^6 - 6Ax^6 + Ax^6 = x^6.$$
Therefore $A = 1/25$, and the particular solution is $g_p = x^6/25$.

Third step: combine the complementary and particular solutions to construct the general solution of the differential equation. It is
$$g = c_1x \; + c_2x\log x + \frac{x^6}{25}.$$

3. Find the general solution to

$$\ddot{f}(t) + \dot{f}(t) - 6f(t) = t.$$

Solution:

First step: find two linearly independent complementary solutions to the homogeneous equation $\ddot{f}(t) + \dot{f}(t) - 6f(t) = 0$. Use the trial function $e^{\lambda t}$. The characteristic equation is $\lambda^2 + \lambda - 6 = 0$, which has two real roots at $\lambda_1 = 2$ and $\lambda_2 = -3$. So the two independent complementary solutions, are

$$f_1 = e^{2t}, \qquad f_2 = e^{-3t}.$$

Second step: find a particular solution to the nonhomogeneous equation. The trial function is a polynomial of degree 1 with undetermined coefficients: $f_p = At + B$. The first and second derivatives of this trial function are $\dot{f}_p = A$ and $\ddot{f}_p = 0$.

Substitute these functions into the differential equation:

$$\begin{aligned} A - 6(At + B) &= t \\ (-6A)t + (A - 6B) &= t \end{aligned}$$

Therefore $-6A = 1$ and $A - 6B = 0$, from which we conclude that $A = -1/6$ and $B = -1/36$. The particular solution is thus $f_p = -t/6 - 1/36$.

Third step: combine the complementary and particular solutions to construct the general solution of the differential equation. It is

$$f(t) = c_1 e^{2t} + c_2 e^{-3t} - \frac{t}{6} - \frac{1}{36}$$

4. Find the solution to

$$\ddot{f}(t) + \dot{f}(t) - 6f(t) = e^{4t}$$

with initial values $f(0) = 0, \dot{f}(0) = 0$.

Solution:

First step: find two linearly independent complementary solutions to the homogeneous equation $\ddot{f}(t) + \dot{f}(t) - 6f(t) = 0$. This is the same equation we considered in the previous problem. As above, the two independent complementary solutions, are

$$f_1 = e^{2t}, \qquad f_2 = e^{-3t}.$$

Second step: find a particular solution to the nonhomogeneous equation. The trial function is a multiple of the exponential function on the right-hand side, $f_p = Ce^{4t}$. The first and second derivatives of this trial function are $\dot{f}_p = 4Ce^{4t}$ and $\ddot{f}_p = 16Ce^{4t}$. Substitute these functions into the differential equation:

$$16Ce^{4t} + 4Ce^{4t} - 6Ce^{4t} = e^{4t}$$

This equation is satisfied for $C = 1/14$. Thus the particular solution is $f_p = e^{4t}/14$.

Third step: combine the complementary and particular solutions to construct the general solution of the differential equation. It is

$$f(t) = c_1 e^{2t} + c_2 e^{-3t} + \frac{1}{14} e^{4t}$$

Fourth step: fix the remaining constants using the initial values. We need the first derivative of $f(t)$, which is

$$\dot{f}(t) = 2c_1 e^{2t} - 3c_2 e^{-3t} + \frac{2}{7} e^{4t}$$

Now we evaluate $f(0)$ and $\dot{f}(0)$ and set them equal to the values given.

$$\begin{aligned} c_1 + c_2 + \frac{1}{14} &= 0 \\ 2c_1 - 3c_2 + \frac{2}{7} &= 0 \end{aligned}$$

We can solve this linear system of equations by routine elimination to find $c_1 = -1/10, c_2 = 1/35$. Therefore the full solution is given by

$$f(t) = -\frac{1}{10}e^{2t} + \frac{1}{35}e^{-3t} + \frac{1}{14}e^{4t}.$$