MA22S3 Tutorial Sheet 7: Solutions

23–24 November 2016

1. Find the general solutions to the following differential equations.

   (a) $\ddot{P}(t) + 4\dot{P}(t) + 4P(t) = 0$.

   (b) $f''(x) - 2f'(x) + 6f(x) = 0$.

   (c) $2\ddot{s}(t) + s(t) = 0$.

   Solution:

   (a) Use a trial solution $P = e^{\lambda t}$. The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$, so there is a double root at $\lambda = -2$. The general solution is then

   $$P(t) = (c_1 + tc_2)e^{-2t}.$$

   (b) Use a trial solution $f = e^{\lambda x}$. The characteristic equation is $\lambda^2 - 2\lambda + 6 = 0$, and the two roots are $\lambda = 1 \pm i\sqrt{5}$. So the general solution to the ODE is

   $$f(x) = c_1 e^{(1+i\sqrt{5})x} + c_2 e^{(1-i\sqrt{5})x}$$

   or alternatively

   $$f(x) = c_3 e^x \cos \sqrt{5}x + c_4 e^x \sin \sqrt{5}x.$$

   (c) Use a trial solution $s = e^{\lambda t}$. The characteristic equation is $2\lambda^2 + 1 = 0$, and the two roots are $\lambda = \pm \frac{i}{\sqrt{2}}$. So the general solution to the ODE is

   $$s(t) = c_1 e^{\frac{i}{\sqrt{2}}t} + c_2 e^{-\frac{i}{\sqrt{2}}t}$$

   or alternatively

   $$s(t) = c_3 \cos \frac{t}{\sqrt{2}} + c_4 \sin \frac{t}{\sqrt{2}}.$$

2. (a) Verify that $y_1 = x^{-1/2}$ is a solution to the differential equation

   $$4x^2 y'' + (4x - 4x^2)y' - (1 + 2x)y = 0. \qquad (1)$$

   Solution:

   $$y_1' = -\frac{1}{2}x^{-3/2}$$
   $$y_1'' = \frac{3}{4}x^{-5/2}$$
   $$4x^2 y_1'' + (4x - 4x^2)y_1' - (1 + 2x)y_1 = \left(3x^{-1/2}\right) - \left(2x^{-1/2} - 2x^{1/2}\right) - \left(x^{-1/2} + 2x^{1/2}\right) = 0.$$

(b) Use the method of reduction of order to find the general solution to equation (1).

Solution: Write $y_2 = uy_1$, and use the differential equation to solve for the function u:

$$\begin{aligned} y_2 &= y_1 u \\ y_2' &= y_1 u' + y_1' u \\ y_2'' &= y_1 u'' + 2y_1' u' + y_1'' u \end{aligned}$$

$$\begin{aligned} 0 &= 4x^2 y_2'' + (4x - 4x^2) y_2' - (1+2x) y_2 \\ &= 4x^2 (y_1 u'' + 2y_1' u' + y_1'' u) + (4x - 4x^2)(y_1 u' + y_1' u) - (1+2x) y_1 u \\ &= (4x^2 y_1) u'' + (8x^2 y_1' + 4x y_1 - 4x^2 y_1) u' + (4x^2 y_1'' + (4x - 4x^2) y_1' - (1+2x) y_1) u \\ &= (4x^2 y_1) u'' + (8x^2 y_1' + 4x y_1 - 4x^2 y_1) u' + (0) u \end{aligned}$$

because y_1 is a solution. Substituting the expressions for y_1 and its derivatives, we get

$$\begin{aligned} 0 &= 4x^{3/2} u'' + (-4x^{1/2} + 4x^{1/2} - 4x^{3/2}) u' \\ 0 &= u'' - u' \end{aligned}$$

We have obtained a linear first-order differential equation for u'. It can be solved by $u' = e^x$. Therefore

$$u = e^x$$

We have put in arbitrary integration constants because we only need a single particular solution that is linearly independent of y_1 (and linear independence is guaranteed by the method). Therefore the solution y_2 is

$$y_2 = uy_1 = x^{-1/2} e^x$$

and the general solution to the differential equation can be written as

$$y = c_1 x^{-1/2} + c_2 x^{-1/2} e^x.$$