MA22S3 Tutorial Sheet 6: Solutions

16–17 November 2016

1. Solve the following initial value problems.

   (a)
$$y^{-2}\frac{dy}{dx} = \log x, \qquad y(1) = 1.$$

   Solution:
   This equation is separable.
$$\begin{aligned} \int y^{-2}\, \frac{dy}{dx}\; dx &= \int \log x\; dx \\ \int y^{-2}\; dy &= \int \log x\; dx \\ -y^{-1} &= x \log x - x + C \\ y &= -\frac{1}{x \log x - x + C} \end{aligned}$$
   Now fix the constant C with the initial value given.
$$\begin{aligned} 1 &= -\frac{1}{0 - 1 + C} \\ C &= 0 \end{aligned}$$
   Therefore the solution is
$$y = -\frac{1}{x \log x - x}.$$

   (b)
$$x\frac{dy}{dx} = y + xe^{-y/x}, \qquad y(1) = 1.$$

   Solution:
   This equation can be made separable by a change of variables. Divide through by x, and let $u = y/x$. Then $y' = u + u'x$. The differential equation becomes
$$\begin{aligned} u + u'x &= u + e^{-u} \\ e^u u' &= = \frac{1}{x} \\ \int e^u\; du &= \int \frac{dx}{x} \\ e^u &= \log x + C \\ u &= \log(\log x + C) \\ y &= x \log(\log x + C) \end{aligned}$$
   Now fix the constant C with the initial value given.
$$\begin{aligned} 1 &= \log(0 + C) \\ C &= e \end{aligned}$$
   Therefore the solution is
$$y = x \log(\log x + e).$$

2. Find the general solution to the following equation. Here α is a constant.

$$t\frac{dx}{dt} + \alpha x = t^2.$$

Check your solution by substituting it into the differential equation.

Solution: This is a linear equation which can be solved with an integrating factor. Write the equation in its standard form as

$$\frac{dx}{dt} + \frac{\alpha}{t}x = t.$$

Notice that α/t is the derivative of $\alpha \log t$, so we can multiply the equation through by the integrating factor $e^{\alpha \log t} = t^\alpha$ to bring the left-hand-side into the form of a total derivative.

$$\begin{aligned} t^\alpha \frac{dx}{dt} + \alpha t^{\alpha-1} x &= t^{\alpha+1} \\ \frac{d}{dt}\left[t^\alpha x\right] &= t^{\alpha+1} \end{aligned}$$

Integrate both sides to get

$$\begin{aligned} t^\alpha x &= \int t^{\alpha+1}\, dt \\ &= \frac{1}{\alpha+2}t^{\alpha+2} + C \end{aligned}$$

So the general solution is

$$x = \frac{1}{\alpha+2}t^2 + Ct^{-\alpha}$$

where C is an undetermined constant.

To check the solution, we differentiate it to obtain

$$\frac{dx}{dt} = \frac{2}{\alpha+2}t - \alpha C t^{-\alpha-1}.$$

Then the left-hand side of the differential equation is

$$\begin{aligned} t\left(\frac{2}{\alpha+2}t - \alpha C t^{-\alpha-1}\right) + \alpha\left(\frac{1}{\alpha+2}t^2 + Ct^{-\alpha}\right) &= \frac{2}{\alpha+2}t^2 - \alpha C t^{-\alpha} + \frac{\alpha}{\alpha+2}t^2 + \alpha C t^{-\alpha} \\ &= t^2, \end{aligned}$$

which is equal to the right-hand side.