MA22S3 Tutorial Sheet 5: Solutions

2–3 November 2016

Formulas:

- Fourier Transform:

$$\mathcal{F}\left[f(t)\right] = \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\ dt\,.$$

- We have computed the Fourier transforms of a few specific functions, including the following:

$$\mathcal{F}\left[e^{-a|t|}\right] \quad = \quad \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\omega^2}, \qquad \text{for } a > 0$$

- Inverse Fourier Transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\ d\omega\,.$$

- Convolution:

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\ du\,.$$

Questions:

1. Evaluate the following integrals.

   (a) $\int_{-\infty}^{\infty} \delta(3x)(x+2)\ dx = \int_{-\infty}^{\infty} (u/3+2)\ \delta(u)\ \ du/3 = 2/3$.
   (Change of variables: $u = 3x$)

   (b) $\int_0^3 \delta(x+1)\ \cos(x)\ dx = 0\,.$
   The only solution to the delta function is at $x = -1$, which lies outside the range of integration.

   (c) $\int_{-1}^{1} \left[\frac{d}{dx}\delta(x)\right]\ x\ dx = \delta(x)x|_{-1}^{1} - \int_{-1}^{1} \delta(x)\ \ dx = -1$.
   (Integration by parts)

2. Compute the convolution $f \star g$ of the following pair of functions.

$$\begin{aligned} f(t) &= t^2 - 2t \\ g(t) &= \delta(t-3) + \delta(t-4) \end{aligned}$$

Solution:

$$\begin{aligned} (f \star g)(t) &= \int_{-\infty}^{\infty} f(u)g(t-u)\ du \\ &= \int_{-\infty}^{\infty} (u^2-2u)\left[\delta(t-u-3)+\delta(t-u-4)\right]\ du \\ &= (t-3)^2 - 2(t-3) + (t-4)^2 - 2(t-4) \\ &= 2t^2 - 18t + 39 \end{aligned}$$

3. What are the Fourier transforms of the following functions?

(a)

$$\frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+t^2}, \qquad \text{for } a > 0$$

Symmetry/duality principle and the result quoted above: $e^{-a|-\omega|} = e^{-a|\omega|}$.

(b)

$$\frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+(t-1)^2}, \qquad \text{for } a > 0$$

Change variables to $u = t - 1$ and use the previous result:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-1)\, e^{-i\omega t}\, dt &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega(u+1)}\, du \\ &= e^{-i\omega} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du \end{aligned}$$

Taking $f(u) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+u^2}$ above, we can use the previous result to get

$$e^{-a|\omega| - i\omega}.$$

(c)

$$\frac{1}{t^2 - 2t + 5}$$

This function is a constant times the previous function upon taking $a = 2$. The constant multiple is $\sqrt{2\pi}/4$. Therefore the Fourier transform is

$$\frac{\sqrt{2\pi}}{4} e^{-2|\omega| - i\omega}$$

(d)

$$\frac{1}{t^2 - t + 1}$$

By now we can guess that we can construct this function from rescalings and shifts of the previous ones. Specifically, we would be satisfied to find constants a, b, c such that

$$\frac{1}{t^2 - t + 1} = c \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + (t-b)^2}$$

Completing the square in the denominator, we see that we should choose $b = 1/2$. Then we need $a = \sqrt{3}/2$ and $c = \sqrt{2\pi}/(2a) = \sqrt{2\pi/3}$ to cancel the overall constant factor. Applying manipulations like the ones above (start with the fact of part (a), then change variables $u = t - b$, and finally multiply through by c), we arrive at

$$\sqrt{\frac{2\pi}{3}} e^{-\frac{\sqrt{3}}{2}|\omega| - \frac{i}{2}\omega}$$