MA22S3 Tutorial Sheet 4: Solutions

26–27 October 2016

Formulas:

- The real Fourier series expansion of a function $f(t)$ of fundamental period L can be written as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{L}\right),$$

  where the coefficients are given by the Euler formulas:

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\, dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi n t}{L}\right)\, dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi n t}{L}\right)\, dt \end{aligned}$$

- Parseval's Theorem: For a function of period L whose real Fourier series expansion is written in the form above, the following equation is true:

$$\frac{1}{L}\int_{t_0}^{t_0+L} f(t)^2\, dt = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

- Fourier Transform:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt\,.$$

Questions:

1. In Tutorial Sheet 2, we computed the Fourier series of the following function:

$$\begin{aligned} f(t) &= t^2 \quad \text{for } |t| < 1\,, \\ f(t) &= f(t+2). \end{aligned}$$

   The following Fourier coefficients were obtained:

$$a_0 = \frac{2}{3}, \qquad a_n = \frac{4(-1)^n}{\pi^2 n^2}, \qquad b_n = 0\,.$$

   Use Parseval's Theorem to evaluate the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\,.$$

   Solution:

It is convenient to take our integration range to be the interval $(-1, 1)$, which has the length of one period, since this is the interval on which the formula t^2 is valid.

On the left-hand side of Parseval's theorem, we have

$$\begin{aligned} \frac{1}{L}\int_{t_0}^{t_0+L} f(t)^2\, dt &= \frac{1}{2}\int_{-1}^{1} t^4\, dt \\ &= \frac{1}{2}\frac{t^5}{5}\bigg|_{-1}^{1} \\ &= \frac{1}{5}\,. \end{aligned}$$

On the right-hand side, we have

$$\begin{aligned} \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) &= \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{\pi^2 n^2}(-1)^n\right)^2 \\ &= \frac{1}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{\pi^4 n^4} \\ &= \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \end{aligned}$$

Setting the two sides equal, we have

$$\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

and thus

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{90}\,.$$

2. Using the definition given above, compute the Fourier transform of $f(t) = te^{-2|t|}$.

Solution:

Separate the integration region according to the piecewise definition of $|t|$.

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} te^{2t}e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} te^{-2t}e^{-i\omega t}\, dt$$

We're integrating functions of the form $te^{\alpha t}$, which we can do using integration by parts:

$$\int te^{\alpha t}\, dt = \frac{1}{\alpha}te^{\alpha t} - \int \frac{1}{\alpha}e^{\alpha t}\, dt = \frac{1}{\alpha}te^{\alpha t} - \frac{1}{\alpha^2}e^{\alpha t} = \frac{\alpha t - 1}{\alpha^2}e^{\alpha t} \qquad (1)$$

In the first term, we substitute $\alpha \to 2 - i\omega$, and in the second term, we substitute $\alpha \to -2 - i\omega$. Then we have

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left[\frac{(2-i\omega)t - 1}{(2-i\omega)^2}e^{(2-i\omega)t}\right]\bigg|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}}\left[\frac{(-2-i\omega)t - 1}{(-2-i\omega)^2}e^{(-2-i\omega)t}\right]\bigg|_{0}^{\infty}$$

The functions approach zero at $t \to \pm\infty$ because of the exponential factors. All that's left are the terms evaluated at $t = 0$.

$$\begin{aligned} \tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\frac{-1}{(2-i\omega)^2} - \frac{1}{\sqrt{2\pi}}\frac{-1}{(2+i\omega)^2} \\ &= \frac{1}{\sqrt{2\pi}}\frac{(2-i\omega)^2 - (2+i\omega)^2}{(2-i\omega)^2(2+i\omega)^2} \\ &= \frac{1}{\sqrt{2\pi}}\frac{-8i\omega}{(4+\omega^2)^2} \end{aligned}$$