MA22S3 Tutorial Sheet 3: Solutions

19-20 October 2016

Formulas:

- Definition of hyperbolic cosine and hyperbolic sine:

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}, \qquad \sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

- The complex Fourier series expansion of a function $f(t)$ of period L can be written as

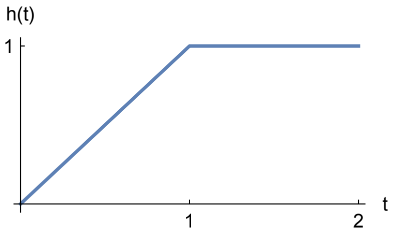
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{L}t},$$

where the coefficients are given by

$$c_n = \frac{1}{L}\int_{t_0}^{t_0+L} f(t) e^{-in\frac{2\pi}{L}t}\, dt.$$

Questions:

1. Write formulas for (a) the simple periodic extension, (b) the half-range even expansion, and (c) the half-range odd expansion of the following function.



Solution:

(a)

$$h_1(t) = \begin{cases} t & \text{for } 0 < t < 1 \\ t & \text{for } 1 \le t < 2 \end{cases}, \qquad \text{and } h_1(t+2) = h_1(t).$$

(b)

$$h_2(t) = \begin{cases} |t| & \text{for } 0 < |t| < 1 \\ 1 & \text{for } 1 \le |t| < 2 \end{cases}, \qquad \text{and } h_2(t+4) = h_2(t).$$

(c)

$$h_3(t) = \begin{cases} t & \text{for } 0 < |t| < 1 \\ 1 & \text{for } 1 \le t < 2 \\ -1 & \text{for } -2 < t \le -1 \end{cases}, \qquad \text{and } h_3(t+4) = h_3(t).$$

2. Compute the complex Fourier series of the following periodic function, and write all the terms with $|n| \leq 3$ explicitly.

$$f(t) = \cosh t \quad \text{for } -\frac{\pi}{2} \leq t < \frac{\pi}{2}, \qquad \text{and } f(t+\pi) = f(t)$$

Solution: The fundamental period is $L = \pi$, and it is convenient to choose the integration range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Therefore

$$\begin{aligned}
c_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\, e^{-2int}\; dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh t\, e^{-2int}\; dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^t + e^{-t}}{2}\, e^{-2int}\; dt \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(1-2in)t} + e^{-(1+2in)t}\; dt \\
&= \frac{1}{2\pi}\left[\frac{1}{1-2in}e^{(1-2in)t} - \frac{1}{1+2in}e^{-(1+2in)t}\right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2\pi}\left[\frac{1}{1-2in}\left(e^{\frac{\pi}{2}-i\pi n} - e^{-\frac{\pi}{2}+i\pi n}\right) - \frac{1}{1+2in}\left(e^{-\frac{\pi}{2}-i\pi n} - e^{\frac{\pi}{2}+i\pi n}\right)\right]
\end{aligned}$$

The expression above is valid for every integer value of n. We substitute $e^{i\pi n} = e^{-i\pi n} = (-1)^n$, which appears in every term, and then simplify by combining the two terms over a common denominator.

$$\begin{aligned}
c_n &= \frac{(-1)^n}{2\pi}\left[\frac{1}{1-2in}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) - \frac{1}{1+2in}\left(e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}\right)\right] \\
&= \frac{(-1)^n\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)}{2\pi}\left[\frac{1}{1-2in} + \frac{1}{1+2in}\right] \\
&= \frac{(-1)^n\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)}{2\pi}\,\frac{2}{1+4n^2}
\end{aligned}$$

We can make one final notational simplification by recognizing that $\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)/2 = \sinh\frac{\pi}{2}$. This is just a numerical factor that appears in each of the coefficients.

So the coefficients are

$$c_n = \frac{2}{\pi}\left(\sinh\frac{\pi}{2}\right)\frac{(-1)^n}{1+4n^2}$$

Finally, we can write the full complex Fourier series as

$$f(t) = \frac{2}{\pi}\sinh\frac{\pi}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+4n^2}e^{2int}$$

and the series starts, for $|n| \leq 3$, with

$$f(t) = \frac{2}{\pi}\sinh\frac{\pi}{2}\left[1 - \frac{1}{5}e^{2it} - \frac{1}{5}e^{-2it} + \frac{1}{17}e^{4it} + \frac{1}{17}e^{-4it} - \frac{1}{37}e^{6it} - \frac{1}{37}e^{-6it} + \cdots\right]$$

3. Convert the complex Fourier series found in the previous problem to a real Fourier series. Check that the coefficients are actually real-valued.

Solution:

We could expand each of the exponentials in terms of a cosine and a sine, but it's not hard to see that they already appear in complex-conjugate pairs combining to form only cosines. Let's separate the constant term and pair up the rest, sorted by magnitude n. All of the coefficients are even functions of n.

$$\begin{aligned}
f(t) &= \frac{2}{\pi}\sinh\frac{\pi}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+4n^2}e^{2int} \\
&= \frac{2}{\pi}\sinh\frac{\pi}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+4n^2}e^{2int}+\frac{(-1)^{-n}}{1+4(-n)^2}e^{2i(-n)t}\right] \\
&= \frac{4}{\pi}\sinh\frac{\pi}{2}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+4n^2}\frac{e^{2int}+e^{-2int}}{2}\right] \\
&= \frac{4}{\pi}\sinh\frac{\pi}{2}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+4n^2}\cos 2int\right]
\end{aligned}$$

Now we have a real Fourier cosine series, with real coefficients. This form is consistent with the fact that our original function $f(t)$ is even in t.