# MA22S3 Tutorial Sheet 2: Solutions

12-13 October 2016

**Formulas:**
The Fourier series expansion of a function $f(t)$ of fundamental period $L$ can be written as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right),$$

where the coefficients are given by the Euler formulas:

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\, dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)\, dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)\, dt \end{aligned}$$

**Questions:**
Define the function $f(t)$ as follows:

$$\begin{aligned} f(t) &= t^2 \quad \text{for } |t| < 1, \\ f(t) &= f(t+2). \end{aligned}$$

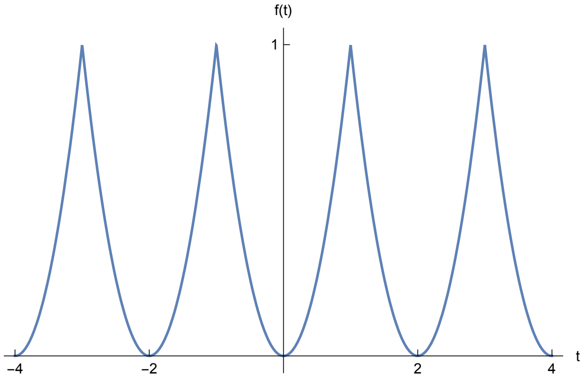
1. What is the fundamental period of $f(t)$?
2. Sketch $f(t)$ on the interval $-4 \leq t \leq 4$.
3. Is $f(t)$ even, odd, or neither?
4. Find the first four terms of the real Fourier series expansion of $f(t)$.
5. Without doing any further integration, sketch the following functions and write the first four terms of their real Fourier series.

$$\begin{aligned} g(t) &= 1 - t^2 \quad \text{for } |t| < 1, \\ g(t) &= g(t+2). \end{aligned}$$

$$\begin{aligned} h(u) &= \frac{u^2}{\pi^2} \quad \text{for } |u| < \pi, \\ h(u) &= h(u+2\pi). \end{aligned}$$

**Solutions:**

1. The fundamental period is $L = 2$.

2. 


3. Even.

4. Since the function is even, the Fourier series contains only cosine terms, and all the sine coefficients are zero (all $b_n = 0$). We can use the formulas provided to obtain the constant term and the cosine coefficients. The fact that the function is even allows us to simplify these integrals as well.

   The constant term is

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(t)\ dt \\ &= 2\int_{0}^{1} f(t)\ dt \\ &= 2\int_{0}^{1} t^2\ dt \\ &= 2\left.\frac{t^2}{3}\right|_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

   The cosine terms have the following coefficients, for $n \geq 1$.

$$\begin{aligned} a_n &= \int_{-1}^{1} f(t)\cos(\pi n t)\ dt \\ &= \int_{-1}^{1} t^2 \cos(\pi n t)\ dt \\ &= 2\int_{0}^{1} t^2 \cos(\pi n t)\ dt \end{aligned}$$

   since the integrand is even. We need to integrate by parts twice:

$$\begin{aligned} a_n &= 2\int_{0}^{1} t^2 \cos(\pi n t)\ dt \\ &= 2\left.\left[\frac{t^2}{\pi n}\sin(\pi n t)\right]\right|_0^1 - 2\int_0^1 \frac{2t}{\pi n}\sin(\pi n t)\ dt \\ &= 0 - \frac{4}{\pi n}\int_0^1 t\sin(\pi n t)\ dt \\ &= -\frac{4}{\pi n}\left(\left.\left[-\frac{t}{\pi n}\cos(\pi n t)\right]\right|_0^1 + \int_0^1 \frac{1}{\pi n}\cos(\pi n t)\ dt\right) \\ &= -\frac{4}{\pi n}\left(-\frac{1}{\pi n}\cos(\pi n) + \frac{1}{(\pi n)^2}\left[\sin(\pi n t)\right]_0^1\ dt\right) \\ &= \frac{4}{\pi^2 n^2}\cos(\pi n) + 0 \\ &= \frac{4}{\pi^2 n^2}(-1)^n \end{aligned}$$

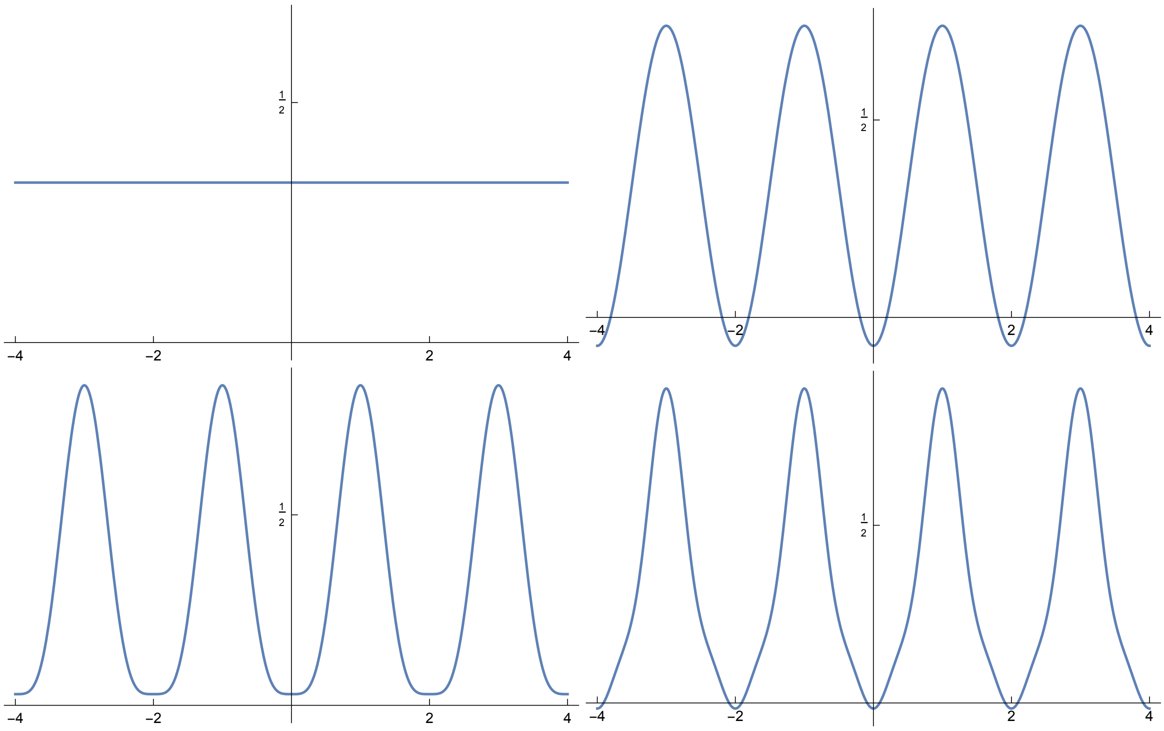
Thus

$$a_0 = \frac{2}{3}, \qquad a_1 = -\frac{4}{\pi^2}, \qquad a_2 = \frac{1}{\pi^2}, \qquad a_3 = -\frac{4}{9\pi^2}$$

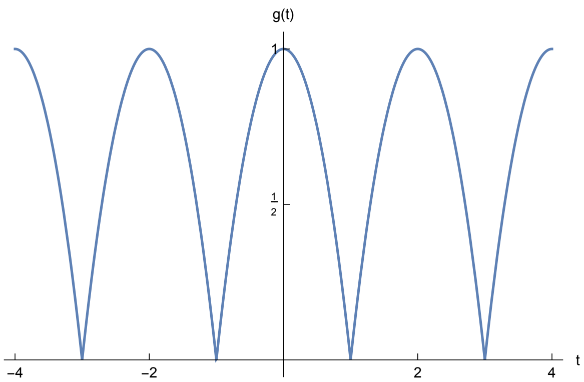
and the first four terms of the Fourier series expansion for $f(t)$ are

$$\frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos(\pi n t) = \frac{1}{3} - \frac{4}{\pi^2}\cos(\pi t) + \frac{1}{\pi^2}\cos(2\pi t) - \frac{4}{9\pi^2}\cos(3\pi t).$$

Here are plots of the first four successive sums in the Fourier series.



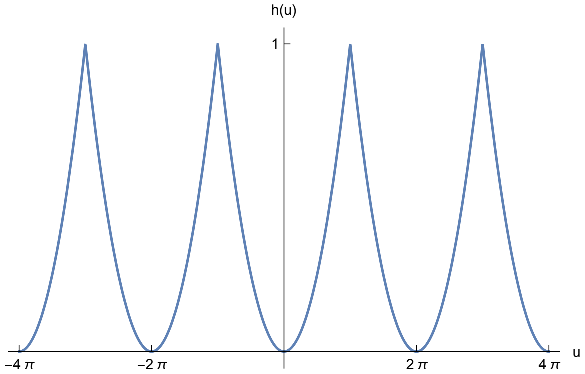

5.




Since $g(t) = 1 - f(t)$, we can subtract the Fourier series obtained above from 1 to write the expansion

$$g(t) = \frac{2}{3} + \frac{4}{\pi^2}\cos(\pi t) - \frac{1}{\pi^2}\cos(2\pi t) + \frac{4}{9\pi^2}\cos(3\pi t) + \cdots$$

We verify that this takes the form of a Fourier series expansion.




Here we can check that the change of variables $t = u/\pi$ maps the function $f(t)$ to the function $h(u)$. (One must be sure to check that the period is transformed correctly.) Let us apply the same change to the Fourier series:

$$h(u) = \frac{1}{3} - \frac{4}{\pi^2}\cos(u) + \frac{1}{\pi^2}\cos(2u) - \frac{4}{9\pi^2}\cos(3u) + \cdots$$

Again, we verify that this expansion takes the form of a Fourier series, so we are done.