MA22S3 Tutorial Sheet 10: Solutions

14–15 December 2016

1. Consider the following differential equation:

$$y'' + x^2 y = 0.$$

(a) Are there any singular points of the differential equation? If so, are they regular singular points? Why or why not?

Solution:
This equation is given in standard form, with manifestly analytic coefficients for y and its derivatives, so there are no singular points.

(b) Write the recursive relation for coefficients in a series solution about the point $x = 0$.

Solution:
The series solution takes the form

$$y = \sum_{m=0}^{\infty} a_m x^m.$$

The second derivative is therefore

$$\begin{aligned} y' &= \sum_{m=0}^{\infty} m a_m x^{m-1} \\ y'' &= \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} \end{aligned}$$

Now we write the terms that appear specifically in the differential equation, and shift the summation index as necessary so that the power of x appears in the same form everywhere, to make it easy to collect equal powers in each term of the series.

$$\begin{aligned} x^2 y &= \sum_{m=0}^{\infty} a_m x^{m+2} \qquad \text{shift } m \to m-4 \\ &= \sum_{m=4}^{\infty} a_{m-4} x^{m-2} \end{aligned}$$

Combining these series into the differential equation, we can write a general sum starting at $m = 2$, but we need to take care of the first few terms individually. The terms with $m = 0$ and $m = 1$ are zero by themselves. So the total series differential equation is

$$0 = 2a_2 + 6a_3 x + \sum_{m=4}^{\infty} (m(m-1)a_m + a_{m-4}) x^{m-2}$$

The recursive relation can therefore be presented as

$$\begin{cases} a_0, a_1 \text{ undetermined;} \\ a_2 = 0, \\ a_3 = 0, \\ a_m = -\frac{1}{m(m-1)} a_{m-4} \quad \text{for } m \geq 4. \end{cases} \tag{1}$$

(It would also be possible to replace the explicit lines for a_2 and a_3 with an extension of the bottom line to $m \geq 0$, along with the definitions $a_{-1} = a_{-2} = 0$.)

2. Let λ be a constant complex number. Consider the following differential equation.

$$(x - x^2)y'' + (1 - x)y' + \lambda y = 0.$$

(a) Are there any singular points of the differential equation? If so, are they regular singular points? Why or why not?

Solution: In standard form, the equation would be written as $y'' + p(x)y' + q(x)y = 0$, where $p(x) = 1/x$ and $q(x) = \lambda/(x(1-x))$. These two functions are analytic everywhere except at 0 and 1, so these are the only singular points. Since $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$, this is a regular singular point. Likewise, $(x-1)p(x)$ and $(x-1)^2q(x)$ are analytic at $x = 1$, so this is also a regular singular point.

(b) Find a general series solution about the point $x_0 = 0$.

Solution: Since x_0 is a regular singular point, we use the Frobenius method: we look for a series solution in the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m+r}.$$

Its derivatives are then

$$\begin{aligned} y_1' &= \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \\ y_1'' &= \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} \end{aligned}$$

Now we write the terms that appear specifically in the differential equation, and shift the summation index as necessary so that the power of x appears everywhere as $m + r$, to make it easy to collect equal powers in each term of the series.

$$\begin{aligned} xy_1'' &= \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-1} \qquad \text{shift } m \to m+1 \\ &= \sum_{m=-1}^{\infty} (m+r+1)(m+r)a_{m+1} x^{m+r} \\ -x^2y_1'' &= \sum_{m=0}^{\infty} -(m+r)(m+r-1)a_m x^{m+r} \\ y_1' &= \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \qquad \text{shift } m \to m+1 \\ &= \sum_{m=-1}^{\infty} (m+r+1)a_{m+1} x^{m+r} \\ -xy_1' &= \sum_{m=0}^{\infty} -(m+r)a_m x^{m+r} \\ \lambda y_1 &= \sum_{m=0}^{\infty} \lambda a_m x^{m+r}. \end{aligned}$$

Substituting these sums back into the differential equation, we have

$$\begin{aligned} 0 &= xy_1'' - x^2y_1'' + y_1' - xy_1' + \lambda y_1 \\ &= r^2 a_0 x^{r-1} + \sum_{m=0}^{\infty} \left[(m+r+1)^2 a_{m+1} - (m+r)^2 a_m + \lambda a_m\right] x^{m+r} \qquad (2) \end{aligned}$$

where the first term is the sum of the terms with $m = -1$ in the series expansions of xy'' and y'. This is the term with the lowest power of x, so setting it equal to zero gives the indicial equation $r^2 = 0$, with a double root at $r = 0$. So we can use this value of r to construct one series solution.

From the terms with $m \geq 0$, and now setting $r = 0$, we conclude that

$$(m+1)^2 a_{m+1} + (\lambda - m^2) a_m = 0$$

The recursive relation for computing coefficients is

$$a_{m+1} = \frac{m^2 - \lambda}{(m+1)^2} a_m \tag{3}$$

The general series solution can now be presented in terms of the result for r and the recursive construction for coefficients, as follows.

$$\begin{cases} y_1 = \sum_{m=0}^{\infty} a_m x^m, \\ a_0 \text{ is undetermined, and} \\ a_{m+1} = \frac{m^2-\lambda}{(m+1)^2} a_m \quad \text{for } m \geq 0. \end{cases} \tag{4}$$

(c) In general, what is the radius of convergence of this solution?

Solution: The radius of convergence about $x = 0$ extends to the nearest singular point, which is $x = 1$. Thus the radius of convergence is 1. It is also straightforward to derive this from the recursive relation above, recalling that the radius of convergence can be obtained from the formula

$$R^{-1} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right|.$$

(d) For which values of λ is the solution a polynomial?

Solution: y_1 will be a polynomial if the series terminates, meaning that the coefficients a_m are all zero above a certain value of m. The only way this can happen for the numerator in the recursive formula, which is $m^2 - \lambda$, to be zero.[1] Since m in equation (3) takes all nonnegative integer values, we get polynomial solutions if λ is the square of an integer.

(e) Write three examples of polynomials satisfying the differential equation as well as the initial condition $y(0) = 1$, together with their required values of λ.

Solution:

The simplest (i.e. lowest-degree) polynomial will be with $\lambda = 0^2 = 0$. Then a_1 and all higher-order coefficients are zero. The solution is simply a_0. With the required initial condition, it is $y_1 = 1$.

Next, let's use $\lambda = 1^2 = 1$. Then $a_1 = -a_0$ while all higher-order coefficients vanish. The solution is $a_0(1-x)$, and the initial condition restricts this to the particular solution $y_1 = 1-x$.

Finally, look at the case with $\lambda = 2^2 = 4$. Then $a_1 = -4a_0$, $a_2 = -\frac{3}{4}a_1 = 3a_0$, and all higher-order coefficients vanish. The solution is $a_0(1 - 4x + 3x^2)$, and the initial condition restricts this to the particular solution $y_1 = 1 - 4x + 3x^2$.

We could have taken as a starting point that the initial condition $y_1(0)$ must be the value of a_0, regardless of the value of λ.

[1] The function $y = 0$ is trivially a polynomial and certainly solves the differential equation, but it isn't a general solution, so it is excluded from this analysis.