# MA22S3 Tutorial Sheet 1: Solutions

5–6 October 2016

1. *Kronecker delta.* Evaluate the following.

   (a) $\delta_{02}$
   (b) $\sum_{i=1}^{\infty} \delta_{ik} a_i$
   (c) $\sum_{i=1}^{\infty} \sum_{j=-\infty}^{\infty} \delta_{ik}\delta_{jl}(-1)^j b_{ii}$
   (d) $\sum_{i=2}^{4} \sum_{j=-1}^{1} \delta_{ij}$
   (e) $\sum_{i=2}^{4} \sum_{j=2}^{3} \delta_{ij}$
   (f) $\sum_{i=2}^{4} \sum_{j=2}^{3} \delta_{ij}(i+j)^2$

   *Solution:* Everything follows directly from the definition of the Kronecker delta,

   $$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

   (a) 0
   (b) $\begin{cases} a_k & \text{if } k \text{ is a positive integer} \\ 0 & \text{otherwise} \end{cases}$
   (c) $\begin{cases} (-1)^l b_{kk} & \text{if } k \text{ is a positive integer and } l \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$
   (d) 0
   (e) 2
   (f) $4^2 + 6^2 = 52$

2. Let $V$ be the vector space of all polynomials in $x$ of degree 2 with real coefficients, with an inner product defined by

   $$\langle p(x)|q(x)\rangle = \int_0^1 p(x)q(x)\ dx.$$

   (a) What is the dimension of $V$?
   (b) Is there a real value of $c$ such that the polynomials $x^2 - cx$ and $x$ are orthogonal?

   *Solution:*

   (a) 3, parametrized for example by the coefficients $a_0, a_1, a_2$ of the second-degree polynomial $a_2x^2 + a_1x + a_0$.
   (b) Yes: $\int_0^1 (x^2 - cx)x\ dx = \frac{1}{4} - \frac{c}{3}$, which takes the value 0 for $c = \frac{3}{4}$.

3. Consider the following three vectors in $\mathbb{R}^3$:

   $$v_1 = (1, 0, 1), \qquad v_2 = (1, \sqrt{6}, -1), \qquad v_3 = (\sqrt{3}, -\sqrt{2}, -\sqrt{3}).$$

(a) Show that $\{v_1, v_2, v_3\}$ is an orthogonal set with respect to the usual dot product.

(b) Because the number of vectors in the set above is equal to the dimension of the vector space, they form a basis. Expand the vector $v = (1, 2, 1)$ in terms of this basis: find real coefficients $a_1, a_2, a_3$ such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

*Solution:*

(a)

$$\begin{aligned} v_1 \cdot v_2 &= 1 + 0 - 1 = 0 \\ v_1 \cdot v_3 &= \sqrt{3} + 0 - \sqrt{3} = 0 \\ v_2 \cdot v_3 &= \sqrt{3} - \sqrt{6}\sqrt{2} + \sqrt{3} = 0 \end{aligned}$$

(b) Compute the inner products of each of the basis vectors with itself:

$$\begin{aligned} v_1 \cdot v_1 &= 1 + 0 + 1 = 2 \\ v_2 \cdot v_2 &= 1 + 6 + 1 = 8 \\ v_3 \cdot v_3 &= 3 + 2 + 3 = 8 \end{aligned}$$

Then,

$$\begin{aligned} a_1 &= \frac{v \cdot v_1}{v_1 \cdot v_1} = 1 \\ a_2 &= \frac{v \cdot v_2}{v_2 \cdot v_2} = \frac{\sqrt{3}}{2\sqrt{2}} \\ a_3 &= \frac{v \cdot v_3}{v_3 \cdot v_3} = -\frac{1}{2\sqrt{2}} \end{aligned}$$

4. Define the following operation on $\mathbb{R}^2$:

$$\langle (x_1, x_2) | (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$$

(a) Show that this operation is symmetric, bilinear, and positive definite. Therefore it defines an inner product on $\mathbb{R}^2$.

(b) With this definition, compute the inner products of the following pairs of vectors:
- $(1, 0)$ and $(0, 1)$
- $(1, 0)$ and $(-2, 1)$
- $(0, 1)$ and $(-2, 1)$

(c) With respect to this same inner product, write an orthonormal set with 2 elements.

*Solution:*

(a)
- Symmetry: $\langle (y_1, y_2) | (x_1, x_2) \rangle = 2y_1^2 + y_1x_2 + y_2x_1 + 2x_2^2 = \langle (x_1, x_2) | (y_1, y_2) \rangle$
- Linearity in the first entry:

$$\begin{aligned} \langle a(w_1, w_2) + b(x_1, x_2) | (y_1, y_2) \rangle &= \langle (aw_1 + bx_1, aw_2 + bx_2) | (y_1, y_2) \rangle \\ &= 2(aw_1 + bx_1)y_1 + ((aw_1 + bx_1))y_2 + (aw_2 + bx_2)y_1 + 2(aw_2 + bx_2)y_2 \\ &= a\left[2w_1y_1 + w_1y_2 + w_2y_1 + 2w_2y_2\right] + b\left[2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2\right] \\ &= a\,\langle (w_1, w_2) | (y_1, y_2) \rangle + b\,\langle (x_1, x_2) | (y_1, y_2) \rangle \end{aligned}$$

  Linearity in the second entry then follows from the symmetry property.
- Positive definiteness: $\langle (x_1, x_2) | (x_1, x_2) \rangle = 2x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_1^2 + x_2^2$ A sum of squares of real numbers is always positive unless each of the squares is separately zero, which in this case happens only if $x_1 = x_2 = 0$, i.e. $(x_1, x_2) = (0, 0)$.

(b) 1, -3, 0.

(c) We see that the last pair is orthogonal, so one easy option is to normalise that set by dividing each element by its norm. $\|(0, 1)\| = \sqrt{2}$ and $\|(-2, 1)\| = \sqrt{6}$, so we can choose the set $\left\{\left(0, \frac{1}{\sqrt{2}}\right), \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$.