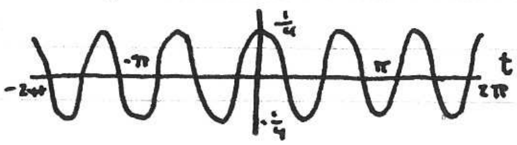
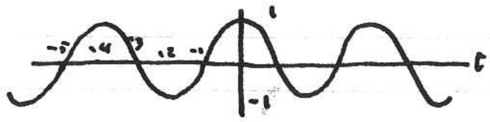
MA22S3 Tutorial 1

1 Sketch in the domain $-2\pi < t < 2\pi$

a $\frac{1}{4}\cos(3t)$

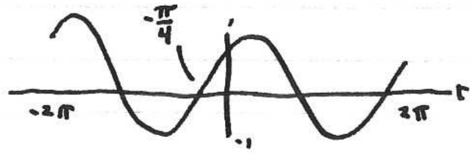


b $\cos\left(\frac{\pi t}{2}\right)$



, $L = 4$

c $\sin\left(t + \frac{\pi}{4}\right)$



2 Let V be the ~~inner product~~ vector space of all polynomials in x of degree 1 with real coefficients, with an inner product defined by

$$\langle p(x) | q(x) \rangle = \int_0^1 dx\, p(x)\, q(x).$$

a The dimension of V, $\dim(V)$, is given by the size of the set of basis vectors. The simplest basis is the monomial basis $\{1, x\}$, so $\dim(V) = 2$

spans, linearly indep.

b Are $3x-1$ and x orthogonal?

$$\int_0^1 dx\,(3x^2 - x) = \left[x^3 - \frac{x^2}{2}\right]_0^1 = \frac{1}{2} \neq 0 \quad \therefore \text{not orthogonal.}$$

3 $v_1 = (1,-1,0)$, $v_2 = (\frac{1}{2}, \frac{1}{2}, 1)$, $v_3 = (-1,-1,1)$ $\in \mathbb{R}^3$.

a $v_1 \cdot v_2 = \frac{1}{2} - \frac{1}{2} = 0$, $v_1 \cdot v_3 = -1 + 1 = 0$, $v_2 \cdot v_3 = 0$ $\therefore$ $\{v_1, v_2, v_3\}$ is an orthogonal set.

b Expand $(2,0,3)$ in terms of this basis.

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3, \quad a_1, a_2, a_3 \in \mathbb{R}.$$

$$a_1 = \frac{v \cdot v_1}{v_1 \cdot v_1} = 1, \quad a_2 = \frac{v \cdot v_2}{v_2 \cdot v_2} = \frac{4}{3/2} = \frac{8}{3},$$

$$a_3 = \frac{v \cdot v_3}{v_3 \cdot v_3} = \frac{1}{3}.$$

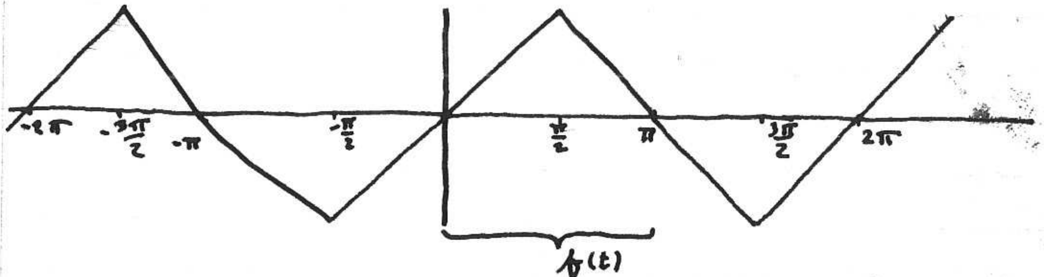
30 + 45 + 60 + 90?
225 = 3~4.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{L}\right), \quad a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\cos\left(\frac{2\pi n t}{L}\right)dt$$

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\sin\left(\frac{2\pi n t}{L}\right)dt.$$

MA22S3 Tutorial 2

1 $$f(t) = \begin{cases} t & \text{for } 0 \le t \le \frac{\pi}{2} \\ \pi - t & \text{for } \frac{\pi}{2} \le t \le \pi \end{cases} \quad \text{(triangle wave).}$$



Odd extension: $f(t) = \begin{cases} t & \text{for } -\frac{\pi}{2} \le t < \frac{\pi}{2} \\ \pi - t & \text{for } \frac{\pi}{2} \le t < \frac{3\pi}{2} \end{cases}$, $f(t+2\pi) = f(t)$

$L = 2\pi$

End of question: $b_1 = \frac{4}{\pi}$, $b_3 = -\frac{4}{9\pi}$, $b_5 = \frac{4}{25\pi}$, $b_7 = -\frac{4}{49\pi}$,
$b_2 = b_4 = b_6 = \dots = 0.$

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n^2} & n = 4k+1 \\ -\frac{4}{\pi n^2} & n = 4k+3 \end{cases} \quad k \in \mathbb{Z}$$

$$f(t) = \frac{4}{\pi}\sin(t) - \frac{4}{9\pi}\sin(3t) + \frac{4}{25\pi}\sin(5t) - \frac{4}{49\pi}\sin(7t) + \dots$$

2 Since f is odd, $a_n = 0 \ \forall n$.

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\sin\left(\frac{2\pi n t}{L}\right)dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$
$$= \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,dt$$
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} t\sin(nt)\,dt + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - t)\sin(nt)\,dt$$

$$\int t\sin(nt)\,dt = uv - \int v\,du \quad \left(u = t,\ v = \int dv = \int \sin(nt)\,dt = -\frac{1}{n}\cos(nt)\right)$$
$$= -\frac{t}{n}\cos(nt) + \int \frac{1}{n}\cos(nt)\,dt$$
$$= -\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) + c.$$

$$\therefore b_n = \frac{2}{\pi}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_0^{\frac{\pi}{2}} - 2\left[\frac{1}{n}\cos(nt)\right]_{\frac{\pi}{2}}^{\pi}$$
$$- \frac{2}{\pi}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_{\frac{\pi}{2}}^{\pi}$$
$$= -\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n}\cos(n\pi) + \frac{2}{n}\cos\left(\frac{n\pi}{2}\right)$$
$$+ \frac{2}{n}\cos(n\pi) - \cancel{\frac{2}{\pi n^2}\sin(n\pi)} - \frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

0+30+120+30

$$= \cos\left(\frac{n\pi}{2}\right)\left(-\frac{1}{n} + \frac{2}{n} - \frac{1}{n}\right) + \cos(n\pi)\left(-\frac{2}{n} + \frac{2}{n}\right) + \sin\left(\frac{n\pi}{2}\right)\left(\frac{2}{\pi n^2} + \frac{2}{\pi n^2}\right)$$
$$= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right).$$

MA22S3 Tutorial 3

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \le t < 1 \end{cases}, \quad f(t+1) = f(t). \qquad [1]$$

$$c_n = \frac{1}{L}\int_{t_0}^{t_0+L} dt\, f(t)\, e^{-\frac{2\pi i n}{L}t}$$

$$= \int_0^1 dt\, f(t)\, e^{-2\pi i n t}$$

$$= \int_0^{\frac{1}{4}} dt\, e^{-2\pi i n t}$$

$$= -\frac{1}{2\pi i n}\left[e^{-2\pi i n t}\right]_0^{\frac{1}{4}} \qquad (n \neq 0)$$

$$= -\frac{1}{2\pi i n}\left[e^{-\frac{\pi}{2} i n} - 1\right] = -\frac{1}{2\pi i n}\left[i^{-n} - 1\right], \qquad [2]$$

$$c_0 = \int_0^{\frac{1}{4}} dt = \frac{1}{4}. \qquad [1]$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n}{L}t} = \frac{1}{4} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2\pi i n}(i^{-n}-1)\, e^{2\pi i n t} \qquad [1]$$

$$= \frac{1}{4} + \frac{1-i}{2\pi}e^{2\pi i t} + \frac{1+i}{2\pi}e^{-2\pi i t} - \frac{i}{2\pi}e^{4\pi i t} + \frac{i}{2\pi}e^{-4\pi i t} - \frac{1+i}{6\pi}e^{6\pi i t} - \frac{1-i}{6\pi}e^{-6\pi i t} + 0 + \cdots \qquad [1]$$

2 $$f(t) = \frac{1}{4} - \sum_{\substack{n\neq 0 \\ n=-\infty}}^{\infty} \frac{1}{2\pi i n}\left(\cos(2\pi n t) + i\sin(2\pi n t)\right)\left(i^{-n}-1\right) \qquad [1]$$

$$= \frac{1}{4} - \sum_{n=1}^{\infty}\frac{1}{2\pi i n}(i^{-n}-1)(\cos(2\pi i n) + i\sin(2\pi i n)) - \sum_{n=-\infty}^{-1}\frac{1}{2\pi i n}(i^{-n}-1)(\quad) \qquad [1]$$

$$\cancel{= \frac{1}{4} - \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i n}(i^{-n}-1) + \frac{1}{2\pi i(-n)}(i^{n}-1)\right)(\cos(2\pi n t) + i\sin(2\pi n t))}$$

$$= \frac{1}{4} - \sum_{n=1}^{\infty}\frac{1}{2\pi i n}(i^{-n}-1)(\cos(2\pi n t) + i\sin(2\pi n t))$$

$$- \sum_{n=1}^{\infty}\frac{1}{2\pi i(-n)}(i^{n}-1)(\cos(-2\pi n t) + i\sin(-2\pi n t))$$

$$= \frac{1}{4} - \sum_{n=1}^{\infty}\Bigg[\frac{1}{2\pi i n}(i^{-n}-1)(\cos(2\pi n t) + i\sin(2\pi n t)) \qquad [1]$$

$$- \frac{1}{2\pi i n}(i^{n}-1)(\cos(2\pi n t) - i\sin(2\pi n t))\Bigg]$$

$$= \frac{1}{4} - \sum_{n=1}^{\infty}\Bigg[\left(\frac{1}{2\pi i n}(i^{-n}-1) - \frac{1}{2\pi i n}(i^{n}-1)\right)\cos(2\pi n t)$$

$$+ \left(\frac{1}{2\pi i n}(i^{-n}-1) + \frac{1}{2\pi i n}(i^{n}-1)\right) i\sin(2\pi n t)\Bigg] \qquad [1]$$

120 + 45 + 50 + 200

$$= \frac{1}{4} - \sum_{n=1}^{\infty} \frac{1}{2\pi i n}\Big[\left[(i^{-n}-1) - (i^{n}-1) \right] \cos(2\pi n t) + i\left[(i^{-n}-1) + (i^{n}-1) \right] \sin(2\pi n t) \Big]$$

$$= \frac{1}{4} - \sum_{n=1}^{\infty} \frac{1}{2\pi i n}\left[(i^{-n}-i^{n}) \cos(2\pi n t) + i(i^{-n}+i^{n}-2) \sin(2\pi n t) \right]$$

which is of the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{L}\right)$$

where

$$a_n = -\frac{1}{2\pi i n}(i^{-n}-i^{n}) = \begin{cases} 0 & \text{if } n \text{ even}, \\ \frac{1}{\pi n} & \text{if } n \text{ odd and } n = 4k+1, \\ -\frac{1}{\pi n} & \text{if } n \text{ odd and } n = 4k+3, \quad k \in \mathbb{Z} \end{cases}$$

and

$$b_n = -\frac{1}{2\pi n}(i^{-n}+i^{n}-2) = \begin{cases} \frac{1}{\pi n} & \text{if } n \text{ odd}, \\ \frac{2}{\pi n} & \text{if } n \text{ even and } n = 4k+2, \\ 0 & \text{if } n \text{ even and } n = 4k, \quad k \in \mathbb{Z}. \end{cases}$$

Or, since $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $\theta = -\frac{\pi}{2}n$, $\sin(-\frac{\pi}{2}n) = \dfrac{i^{-n} - i^{n}}{2i}$,

$$i^{-n} - i^{n} = -2i \sin\left(\tfrac{\pi}{2}n\right), \quad a_n = \frac{1}{\pi n}\sin\left(\tfrac{\pi}{2}n\right) \in \mathbb{R}.$$

Similarly, $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\theta = -\frac{\pi}{2}$, $\cos(-\frac{\pi}{2}) = \dfrac{i^{-n}+i^{n}}{2}$,

$$i^{-n}+i^{n} = 2\cos\left(\tfrac{\pi}{2}n\right), \quad b_n = -\frac{1}{\pi n}\cos\left(\tfrac{\pi}{2}n\right) + \frac{1}{\pi n} = \frac{1}{\pi n}\left(1 - \cos\left(\tfrac{\pi}{2}n\right)\right) \in \mathbb{R}.$$

MA22S3 Tutorial 4

1 $$f(t)=\begin{cases}\sin(t) & 0<t<\pi\\ 0 & \text{otherwise}\end{cases}$$

$$\tilde f(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\; f(t)e^{-i\omega t}$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\pi}dt\;\sin(t)\,e^{-i\omega t} \qquad [1]$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\pi}dt\;\frac{e^{it}-e^{-it}}{2i}e^{-i\omega t} \qquad [1]$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_0^{\pi}dt\left(e^{it-i\omega t}-e^{-it-i\omega t}\right)$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_0^{\pi}dt\left(e^{i(1-w)t}-e^{i(-1-w)t}\right)$$

$$=\frac{1}{2i\sqrt{2\pi}}\left[\frac{1}{i(1-w)}e^{i(1-w)t}+\frac{1}{i(1+w)}e^{i(-1-w)t}\right]_0^{\pi} \qquad [1]$$

$$=\frac{1}{2i\sqrt{2\pi}}\left(\frac{1}{i-iw}e^{i\pi-i\pi w}+\frac{1}{i+iw}e^{-i\pi-i\pi w}-\frac{1}{i-iw}-\frac{1}{i+iw}\right)$$

$$=\frac{1}{2i\sqrt{2\pi}}\left(\frac{1}{i-iw}e^{i\pi}e^{-i\pi w}+\frac{1}{i+iw}e^{-i\pi}e^{-i\pi w}-\frac{1}{i-iw}-\frac{1}{i+iw}\right)$$

$$=\frac{1}{2i\sqrt{2\pi}}\left(\frac{-1}{i-iw}e^{-i\pi w}+\frac{-1}{i+iw}e^{-i\pi w}-\frac{1}{i-iw}-\frac{1}{i+iw}\right) \qquad [1]$$

$$=\frac{-i}{2\sqrt{2\pi}}\left(\frac{-1}{i-iw}e^{-i\pi w}+\frac{-1}{i+iw}e^{-i\pi w}-\frac{1}{i-iw}-\frac{1}{i+iw}\right)$$

$$=\frac{1}{2\sqrt{2\pi}}\left(\frac{1}{1-w}e^{-i\pi w}+\frac{1}{1+w}e^{-i\pi w}+\frac{1}{1-w}+\frac{1}{1+w}\right)$$

$$=\frac{1}{2\sqrt{2\pi}}\left(e^{-i\pi w}+1\right)\left(\frac{1}{1-w}+\frac{1}{1+w}\right) \qquad [1]$$

$$=\frac{1}{2\sqrt{2\pi}}\left(e^{-i\pi w}+1\right)\cdot\frac{2}{1-w^2} \qquad [1]$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{-i\pi w}+1}{1-w^2}.$$

2 Show that $|\tilde f(w)|=\sqrt{\frac{2}{\pi}}\left|\frac{\cos\left(\frac{\pi w}{2}\right)}{1-w^2}\right|$.

$$|\tilde f(w)|^2=\tilde f(w)^*\tilde f(w)$$

$$=\frac{1}{2\pi}\frac{(e^{i\pi w}+1)(e^{-i\pi w}+1)}{(1-w^2)^2} \qquad [1]$$

70 + 30 + 60 + 60

$$= \frac{1}{2\pi} \cdot \frac{e^{+i\pi w - i\pi w} + e^{i\pi w} + e^{-i\pi w} + 1}{(1-w^2)^2}$$

$$= \frac{1}{2\pi} \cdot \frac{2 + e^{i\pi w} + e^{-i\pi w}}{(1-w^2)^2} \qquad [1]$$

$$= \frac{1}{2\pi} \cdot \frac{2 + 2\cos(\pi w)}{(1-w^2)^2} \qquad [1]$$

$$= \frac{1}{\pi} \cdot \frac{1 + \cos(\pi w)}{(1-w^2)^2}$$

$$= \frac{1}{\pi} \cdot \frac{2\cos^2(\frac{\pi w}{2})}{(1-w^2)^2}, \quad (\cos(2\theta) = 2\cos^2(\theta) - 1) \qquad [1]$$

$$|\tilde{f}(w)| = \sqrt{\frac{2}{\pi}} \left| \frac{\cos(\frac{\pi w}{2})}{1-w^2} \right| .$$

3 $$g(t) = \left\{ \begin{array}{ll} \cos(t) & \cancel{\text{for}}\ \frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{array} \right\} = f(t + \tfrac{\pi}{2}),$$

$$\tilde{g}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t + \tfrac{\pi}{2})\, e^{-iwt}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, f(u)\, e^{-iw(u - \frac{\pi}{2})}$$

$$= e^{i\frac{\pi}{2}w} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, f(u)\, e^{-iwu} \qquad [1]$$

$$= e^{i\frac{\pi}{2}w} \tilde{f}(w)$$

$$= \frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{2}w} \frac{e^{-i\pi w} + 1}{1-w^2} \qquad [1]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-i\frac{\pi}{2}w} + e^{i\frac{\pi}{2}w}}{1-w^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2\cos(\frac{\pi}{2}w)}{1-w^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\cos(\frac{\pi w}{2})}{1-w^2}$$

$(f*g)(t) = \int_{-\infty}^{\infty} du\, f(u)\, g(t-u)$

MA22S3 Tutorial 5

1 a $$\int_0^2 dt\, \delta(t+1)\, t^3 = 0. \quad (t=-1 \notin [0,2])$$

b $$\int_{-\infty}^{\infty} dt \left[\delta(t-2) + \delta(t)\right] e^t = e^2 + 1.$$

c $$\int_{-1}^{1} dt\, \delta(2t)(t-1) = \frac{1}{2}\int_{-2}^{2} du\, \delta(u)\left(\frac{u}{2}-1\right)$$
$$= \frac{1}{2}\int_{-2}^{2} du\, \delta(u)\frac{u}{2} - \frac{1}{2}\int_{-2}^{2} du\, \delta(u)$$
$$= 0 - \frac{1}{2} = -\frac{1}{2}.$$ [3]

2 $$f(t) = \begin{cases} \sin(t) & 0<t<\pi \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\pi\omega}+1}{1-\omega^2}.$$

$(\mathcal{F}^{-1} = R\mathcal{F} = \mathcal{F}R)$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \frac{1}{\sqrt{2\pi}} \frac{e^{-i\pi\omega}+1}{1-\omega^2} e^{i\omega t}, \quad g(t) = \frac{e^{-i\pi t}+1}{1-t^2},$$

$$f(-t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \frac{1}{\sqrt{2\pi}} \underbrace{\frac{e^{-i\pi\omega}+1}{1-\omega^2}}_{g(\omega)} e^{-i\omega t}$$
$$= \frac{1}{\sqrt{2\pi}} \tilde{g}(t),$$

$$\tilde{g}(t) = \sqrt{2\pi}\, f(-t), \quad \tilde{g}(w) = \sqrt{2\pi}\, f(-w) = \begin{cases} -\sqrt{2\pi}\sin(w) & -\pi<w<0 \\ 0 & \text{otherwise}. \end{cases}$$ [4]

3 $$h(t) = \begin{cases} 1 & 0<t<1 \\ 0 & \text{otherwise}. \end{cases}$$

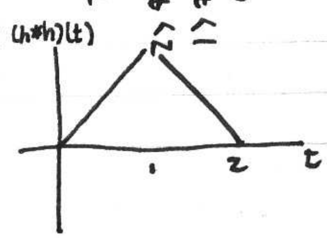
$$(h*h)(t) = \int_{-\infty}^{\infty} h(u)\, h(t-u)\, du$$
$$= \int_0^1 h(t-u)\, du$$ $= 0$ unless $0<t-u<1 \sim -t<-u<1-t$
$\sim t-1<u<t.$

$t<0 \Rightarrow u<0$, $t>2 \Rightarrow 1<u$ $\therefore$ integrand zero. For $0<t<1$,

$$\int_0^1 du\, h(t-u) = \int_0^t du\, h(t-u) + \int_t^1 du\, h(t-u)$$
$$= \int_0^t du + \int_t^1 du\, 0 = t.$$

$1<t<2$: $$\int_0^1 du\, h(t-u) = \int_0^{t-1} du\, h(t-u) + \int_{t-1}^1 du\, h(t-u)$$
$$= \int_0^{t-1} du\, 0 + \int_{t-1}^1 du = 2-t.$$

$$(h*h)(t) = \begin{cases} 0 & t<0 \\ t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$$



[5]

40 + ~~40~~ 120 + 150

MA22S3 Tutorial ~~8~~ six

1 a $$y' + (x+1)y^2 = 0, \quad y'(0) = 1,$$

$$-\frac{y'}{y^2} = x+1,$$

$$-\int \frac{1}{y^2}\,dy = \int (x+1)\,dx,$$

$$\frac{1}{y} = \frac{x^2}{2} + x + c,$$

$$y = \frac{1}{\frac{x^2}{2} + x + c}$$

$$= \frac{2}{x^2 + 2x + c} \qquad (c \to 2c).$$

From the ODE,

$$y'(0) + y(0)^2 = 0, \quad y(0)^2 = -1$$

but $y(0) = \frac{2}{c} \Rightarrow \cancel{c = \pm 2}. \quad c = \pm 2i$

$$y = \frac{2}{x^2 + 2x \pm 2i}.$$

b $$xy' = (y-x)^3 + y, \quad y(1) = \frac{3}{2}.$$

$$y' = \frac{1}{x}(y-x)^3 + \frac{y}{x}.$$

Let $u = \frac{y}{x}$, $u' = \frac{d}{dx}(yx^{-1}) = -yx^{-2} + x^{-1}y'$

$$u'x = -\frac{y}{x} + y'$$

$$= -u + y' \Rightarrow y' = u + u'x.$$

$$u + u'x = \frac{x^2}{x^3}(y-x)^3 + \frac{y}{x}$$

$$= (u-1)^3 x^2 + u,$$

$$u'x = (u-1)^3 x^2$$

$$\frac{u'}{(u-1)^3} = x,$$

$$\int \frac{du}{(u-1)^3} = \int dx\, x,$$

$$-\frac{1}{2} \cdot \frac{1}{(u-1)^2} = \frac{x^2}{2} + c,$$

$$\frac{1}{(u-1)^2} = -x^2 + c,$$

$$u - 1 = \pm\sqrt{\frac{1}{-x^2+c}},$$

$$u = 1 \pm \sqrt{\frac{1}{-x^2+c}},$$

$$\frac{y}{x} = 1 \pm \sqrt{\frac{1}{-x^2+c}}, \quad y = x \pm x\sqrt{\frac{1}{-x^2+c}}$$

$$y(1) = 1 \pm \sqrt{\frac{1}{c-1}} = \frac{3}{2}.$$

Since $\sqrt{\frac{1}{c-1}} > 0$ and $\frac{3}{2} > 1$ we must take $\oplus$ root.

$$1 + \sqrt{\frac{1}{c-1}} = \frac{3}{2} \implies c = \cancel{4}5 \quad \left(1 + \sqrt{\frac{1}{4}} = 1 + \frac{1}{2} = \frac{3}{2} \right).$$

$$y = \cancel{2}\, x + x\sqrt{\frac{1}{5-x^2}}.$$

2 $$L\frac{dI}{dt} + RI = \sin(\omega t), \quad R, L, \omega \text{ constants.}$$

Standard form: $\frac{dI}{dt} + \frac{R}{L} I = \frac{1}{L}\sin(\omega t)$.

Integrating factor $e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$, so

$$e^{\frac{R}{L}t}\frac{dI}{dt} + e^{\frac{R}{L}t}\frac{R}{L} I = \frac{1}{L} e^{\frac{R}{L}t}\sin(\omega t),$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t} I\right) = \frac{1}{L} e^{\frac{R}{L}t}\sin(\omega t)$$

$$e^{\frac{R}{L}t} I = \frac{1}{L}\int dt\, e^{\frac{R}{L}t}\sin(\omega t) + c$$

$$= \frac{1}{2Li}\int e^{\frac{R}{L}t}\left(e^{i\omega t} - e^{-i\omega t}\right) dt \quad + c$$

$$= \frac{1}{2Li}\int dt\left(e^{(\frac{R}{L}+i\omega)t} - e^{(\frac{R}{L}-i\omega)t}\right) + c$$

$$= \frac{1}{2Li}\left(\frac{1}{\frac{R}{L}+i\omega}e^{\frac{R}{L}t+i\omega t} - \frac{1}{\frac{R}{L}-i\omega}e^{\frac{R}{L}t-i\omega t}\right) + c,$$

$$I = \frac{1}{2Li}\left(\frac{1}{\frac{R}{L}+i\omega}e^{i\omega t} - \frac{1}{\frac{R}{L}-i\omega}e^{-i\omega t}\right) + ce^{-\frac{R}{L}t}$$ [5]

$$= \frac{1}{2}\left(\frac{1}{iR-\omega L}e^{i\omega t} - \frac{1}{iR+\omega L}e^{-i\omega t}\right) + ce^{-\frac{R}{L}t}$$

$$= \frac{1}{2}\left(\frac{iR+\omega L}{-R^2-\omega L^2}e^{i\omega t} - \frac{iR-\omega L}{-R^2-\omega L^2}e^{-i\omega t}\right) + ce^{-\frac{R}{L}t}$$

$$= \frac{1}{2}\cdot\frac{1}{-R^2-\omega L^2}\left(iRe^{i\omega t} - iRe^{-i\omega t} + \omega Le^{i\omega t} + \omega Le^{-i\omega t}\right) + ce^{-\frac{R}{L}t}$$

$$= \frac{1}{-R^2-\omega L^2}\left(-R\frac{e^{i\omega t}-e^{-i\omega t}}{2i} + \omega L\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right) + ce^{-\frac{R}{L}t}$$

$$= \frac{1}{-R^2-\omega L^2}\left(-R\sin(\omega t) + \omega L\cos(\omega t)\right) + ce^{-\frac{R}{L}t}$$

$$= \frac{R\sin(\omega t) - \omega L\cos(\omega t)}{R^2+\omega^2L^2} + ce^{-\frac{R}{L}t}.$$

MA2253 Tutorial 7

1 Verify that $y_1 = e^{x^2}$ is a solution to

$$xy'' - y' - 4x^3 y = 0.$$

$$y_1' = 2xe^{x^2}, \quad y_1'' = 4x^2 e^{x^2} + 2e^{x^2},$$

$$xy_1'' - y_1' - 4x^3 y_1 = 4x^3 e^{x^2} + 2xe^{x^2} - 2xe^{x^2} - 4x^3 e^{x^2} = 0.$$ [2]

2 Reduction of order, let $y_2 = uy_1$ and use ODE to solve for u.

$$y_2 = e^{x^2} u, \quad y_2' = e^{x^2} u' + 2xe^{x^2} u,$$

$$y_2'' = e^{x^2} u'' + 2xe^{x^2} u' + 2x \frac{d}{dx} e^{x^2} u + 2e^{x^2} u$$

$$= e^{x^2} u'' + 2xe^{x^2} u' + 2x(e^{x^2} u' + 2xe^{x^2} u) + 2e^{x^2} u$$

$$= e^{x^2} u'' + 4xe^{x^2} u' + (4x^2 + 2) e^{x^2} u.$$

$$xy_2'' - y_2' - 4x^3 y_2 = e^{x^2}(xu'' + 4x^2 u' + 4x^3 u + 2xu$$

$$- u' - 2xu - 4x^3 u)$$

$$= e^{x^2}(xu'' + 4x^2 u' - u') = 0,$$

$$xu'' + 4x^2 u' - u' = 0 \Rightarrow xu'' + u'(4x^2 - 1) = 0,$$

$$xu'' = (1 - 4x^2) u',$$

$$\frac{u''}{u'} = \frac{1}{x} - 4x.$$

$$\int \frac{u''}{u'} dx = \int \frac{1}{s} ds = \log(s) + c \quad (s = u'),$$

$$\log(u') = \log(x) - 2x^2$$

$$u' = xe^{-2x^2}, \quad u = \int -\frac{1}{4} e^s ds = -\frac{1}{4} e^s$$

$$= -\frac{1}{4} e^{-2x^2}.$$

$$y_2 = uy_1 = -\frac{1}{4} e^{-x^2}$$

$$y = c_1 e^{x^2} + c_2 e^{-x^2}.$$

3 $\ddot{f}(t) - 2\dot{f}(t) + 2f(t) = 0 \quad , \quad f(0) = 0, \quad \dot{f}(0) = 1.$

Ansatz: $f(t) = e^{\lambda t}$. Then $\lambda^2 - 2\lambda + 2 = 0$,

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i,$$

$$f(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^t e^{it} + c_2 e^t e^{-it}$$

$$= e^t\left(c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t))\right)$$

$$= e^t\left((c_1+c_2)\cos(t) + i(c_1-c_2)\sin(t)\right) = e^t(c_1\cos(t) + c_2\sin(t)).$$

$$\dot{f}(t) = e^t\left(-c_1\sin(t) + c_2\cos(t) + c_1\cos(t) + c_2\sin(t)\right)$$

$$= e^t\left((c_1+c_2)\cos(t) + (c_2-c_1)\sin(t)\right)$$

$\dot{f}(0) = (c_1 + c_2) = 1$, $f(0) = c_1 = 0 \Rightarrow c_2 = 1$,

$f(t) = e^t \sin(t)$.

4. $\ddot{r}(t) + \dot{r}(t) + \frac{1}{4}r(t) = 0, \quad r(1) = 1, \quad \dot{r}(1) = 0.$

$\lambda^2 + \lambda + \frac{1}{4} = 0$, $\lambda = -\frac{1}{2}$ (degenerate eigenvalue),

$r(t) = (c_1 + c_2 t)e^{-\frac{1}{2}t}$,

$\dot{r}(t) = \left(-\frac{1}{2}c_1 + c_2 - \frac{1}{2}c_2 t\right)e^{-\frac{1}{2}t}$.

$r(1) = (c_1 + c_2)e^{-\frac{1}{2}} = 1$, $\dot{r}(1) = \left(-\frac{1}{2}c_1 + c_2 - \frac{1}{2}c_2\right)e^{-\frac{1}{2}} = 0$,

$-c_1 + c_2 = 0 \Rightarrow c_1 = c_2$,

$\therefore 2c_1 e^{-\frac{1}{2}} = 1$, $c_1 e^{-\frac{1}{2}} = \frac{1}{2}$, $c_1 = \frac{1}{2}e^{\frac{1}{2}}$,

$r(t) = \frac{1}{2}(1+t)e^{\frac{1}{2}(1-t)}$.

MA22S3 Tutorial 8

1 a $$9z^2\frac{d^2g}{dz^2} + 21z\frac{d^2g}{dz^2} + 4g = 0.$$

Euler-Cauchy, so ansatz is $g(z) = z^m$. Characteristic equation is

$$9m^2 + 12m + 4 = 0 \quad \Rightarrow m = -\frac{2}{3} \quad \text{(degenerate eigenvalue)},$$

$$g = c_1 z^{-\frac{2}{3}} + c_2 \log(z) z^{-\frac{2}{3}}.$$

b Let $g_1 = c_{11}z^{-\frac{2}{3}} + c_{12}\log(z)z^{-\frac{2}{3}}$, $g_2 = c_{21}z^{-\frac{2}{3}} + c_{22}\log(z)z^{-\frac{2}{3}}$,

$$\frac{d}{dz}z^{-\frac{2}{3}} = -\frac{2}{3}z^{-\frac{5}{3}}, \quad \frac{d}{dz}\log(z)z^{-\frac{2}{3}} = -\frac{2}{3}\log(z)z^{-\frac{5}{3}} + z^{-\frac{5}{3}}.$$

$$W(g_1, g_2) = \det\begin{pmatrix} g_1 & g_2 \\ \frac{dg_1}{dz} & \frac{dg_2}{dz}\end{pmatrix} = \cancel{\det\begin{pmatrix} c_{11}z^{-\frac{2}{3}} + c_{12}\log(z)z^{-\frac{2}{3}} \\ -c_{11}\frac{2}{3}z^{-\frac{5}{3}} - \frac{2}{3}c_{12}\log(z)z^{-\frac{5}{3}} + c_{12}z^{-\frac{5}{3}}\end{pmatrix}}$$

$$= \det\begin{pmatrix} z^{-\frac{2}{3}}(c_{11} + c_{12}\log(z)) & z^{-\frac{2}{3}}(c_{21} + c_{22}\log(z)) \\ z^{-\frac{5}{3}}\left(-\frac{2}{3}c_{11} + c_{12}\left(-\frac{2}{3}\log(z)+1\right)\right) & z^{-\frac{5}{3}}\left(-c_{21}\frac{2}{3} + c_{22}\left(-\frac{2}{3}\log(z)+1\right)\right)\end{pmatrix}$$

$$= (c_{11} + c_{12}\log(z))\cdot z^{-\frac{7}{3}}\cdot\left(-\frac{2}{3}c_{21} + c_{22}\left(-\frac{2}{3}\log(z)+1\right)\right)$$

$$- (c_{21} + c_{22}\log(z))\cdot z^{-\frac{7}{3}}\cdot\left(-\frac{2}{3}c_{11} + c_{12}\left(-\frac{2}{3}\log(z)+1\right)\right)$$

$$= \Big[\Big(-\frac{2}{3}c_{11}c_{21} + c_{11}c_{22}\left(-\frac{2}{3}\log(z)+1\right) - \frac{2}{3}c_{12}c_{21}\log(z) + c_{12}c_{22}\log(z)\left(-\frac{2}{3}\log(z)+1\right)\Big)$$

$$-\Big(-\frac{2}{3}c_{11}c_{21} + c_{21}c_{22}\left(-\frac{2}{3}\log(z)+1\right) - \frac{2}{3}c_{11}c_{22}\log(z) + c_{12}c_{22}\log(z)\left(-\frac{2}{3}\log(z)+1\right)\Big)\Big]z^{-\frac{7}{3}}$$

$$= \Big[-\frac{2}{3}c_{11}c_{21} + \frac{2}{3}c_{11}c_{21} + c_{11}c_{22}\left(-\frac{2}{3}\log(z)+1\right) + \frac{2}{3}c_{11}c_{22}\log(z) - \frac{2}{3}c_{12}c_{21}\log(z)$$

$$- c_{12}c_{21}\left(-\frac{2}{3}\log(z)+1\right) + c_{12}c_{22}\log(z)\left(-\frac{2}{3}\log(z)+1\right) - c_{12}c_{22}\log(z)\left(-\frac{2}{3}\log(z)+1\right)\Big]z^{-\frac{7}{3}}$$

$$= (c_{11}c_{22} - c_{12}c_{21})z^{-\frac{7}{3}}.$$

2 a $\qquad f''(x) - 2f'(x) - 3f(x) = 3xe^{-x}$ (undetermined coefficients).

Homogeneous case: $f''(x) - 2f'(x) - 3f(x) = 0$. Ansatz $f(x) = e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \quad \lambda_1 = 3,\ \lambda_2 = -1 \quad \Rightarrow f_1 = e^{3x},\ f_2 = e^{-x}.$$

Let $f_p = (Ax + B)xe^{-x} = Ax^2e^{-x} + Bxe^{-x}$.

$$f_p' = -Ax^2e^{-x} + 2Axe^{-x} - Bxe^{-x} + Be^{-x} = e^{-x}(-Ax^2 + B + (2A-B)x),$$

$$f_p'' = -2Axe^{-x} + Ax^2e^{-x} + 2Ae^{-x} - 2Axe^{-x} - Be^{-x} + Bxe^{-x} - Be^{-x}$$

$$= Ax^2e^{-x} + (B-4A)xe^{-x} + (2A-2B)e^{-x}.$$

$$f_p'' - 2f_p' - 3f_p = Ax^2e^{-x} + (B-4A)xe^{-x} + (2A-2B)e^{-x} + 2Ax^2e^{-x} - 2Be^{-x}$$

$$- \cancel{4}2(2A-B)xe^{-x} - 3Ax^2e^{-x} - 3Bxe^{-x}$$

$$= 0x^2e^{-x} + (-8A)xe^{-x} + (2A-4B)e^{-x} \overset{!}{=} 3xe^{-x}$$

$\therefore A = -\frac{3}{8}$, $-\frac{6}{8} - 4B = 0$, $B = -\frac{6}{32} = -\frac{3}{16}$. Then

$$f_p = -\frac{3}{8}x^2e^{-x} - \frac{3}{16}xe^{-x}, \quad f = f_c + f_p$$

$$= c_1e^{3x} + c_2e^{-x} - \frac{3}{8}x^2e^{-x} - \frac{3}{16}xe^{-x}.$$

b $\qquad 4f'' + \frac{65}{x^2}f - 1 = 0 \quad \Leftrightarrow \quad 4x^2f'' + 65f = x^2$

~~Ansatz~~ Homogeneous case: $4x^2f'' + 65f = 0$. Ansatz $f(x) = x^m$

$$4m(m-1) + 65 = 0 \Rightarrow m_\pm = \tfrac{1}{2} \pm 4i,\ x^{m_\pm} = \sqrt{x}\,x^{\pm 4i} = \sqrt{x}\,e^{\pm 4i\log(x)}$$

so $f_c = c_1\sqrt{x}\cos(4\log(x)) + c_2\sqrt{x}\sin(4\log(x))$. Let $f_p = Ax^2$. Then

$$8Ax^2 + 65Ax^2 = x^2, \quad A = \frac{1}{73}, \quad f_p = \frac{x^2}{73},$$

$$f = f_c + f_p = c_1\sqrt{x}\cos(\log(x)) + c_2\sqrt{x}\sin(\log(x)) + \frac{x^2}{73}.$$

MA22S3 Tutorial 9

1 a $$(x+1)^2 y'' + (x+1)y' - y = (x+1)^2 e^x.$$

The homogeneous case is Euler-Cauchy with independent variable $x+1$, so the ansatz is $y(x) = (x+1)^m$.

$$m(m-1) + m - 1 = 0, \quad m^2 - 1 = 0, \quad m = \pm 1,$$

$$y_1 = x+1, \quad y_2 = \frac{1}{x+1}, \quad y_c = c_1 y_1 + c_2 y_2.$$

For variation of parameters, let

$$y_p = u_1 y_1 + u_2 y_2 = u_1 (x+1) + \frac{u_2}{x+1},$$

$$y_p' = u_1'(x+1) + u_1 + \frac{u_2'}{x+1} - \frac{u_2}{(x+1)^2}.$$

$u_i = \int \frac{W_i(y_1,y_2)}{W(y_1,y_2)}\,dx$, W_i is W with i^{th} column replaced by $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix}$.

$W(y_1,y_2) = -\frac{2}{x+1}$, ~~$W_1 = -(x+1)e^x$, $W_2 = (x+1)^3 e^x$~~

$W_1 = -\frac{e^x}{x+1}$, $W_2 = e^x (x+1)$

Our constraint is

$$u_1'(x+1) + \frac{u_2'}{x+1} = 0,$$

so y_p'' is

$$y_p'' = u_1' - \frac{u_2'}{(x+1)^2} + \frac{2u_2}{(x+1)^3}.$$

The ODE gives

$$(x+1)^2 u_1' - u_2' + \frac{2u_2}{x+1} + (x+1)^2 u_1' + (x+1)u_1 + u_2' - \frac{u_2}{x+1} - (x+1)u_1 - \frac{u_2}{x+1}$$

$$= 2(x+1)^2 u_1' \overset{!}{=} (x+1)^2 e^x,$$

$$u_1' = \frac{e^x}{2} \quad \Rightarrow \quad u_2' = -\frac{1}{2}(x+1)^2 e^x.$$

$$u_1 = \frac{1}{2} e^x, \quad u_2 = -\frac{1}{2}\int (x+1)^2 e^x\,dx = -\frac{1}{2}\left[(x+1)^2 e^x - 2\int (x+1)e^x\,dx\right]$$

$$= -\frac{1}{2}\left[(x+1)^2 e^x - 2\left((x+1)e^x - \int e^x\,dx\right)\right]$$

$$= -\frac{1}{2}\left[(x+1)^2 e^x - 2(x+1)e^x + 2e^x\right]$$

2

$$y' + xy = 1, \quad y(0) = 1.$$

$$y = \sum_{m=0}^{\infty} a_m x^m, \quad y' = \sum_{m=0}^{\infty} m a_m x^{m-1}$$

$$= \sum_{m+1=0}^{\infty} (m+1) a_{m+1} x^m$$

$$= \sum_{m=-1}^{\infty} (m+1) a_{m+1} x^m$$

$$= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m.$$

$$xy = \sum_{m=0}^{\infty} a_m x^{m+1} = \sum_{m=1}^{\infty} a_{m-1} x^m.$$

$$y' + xy = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m$$

$$= a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m$$

$$= a_1 + \sum_{m=1}^{\infty} \left((m+1) a_{m+1} + a_{m-1}\right) x^m$$

$$\stackrel{!}{=} 1, \quad \therefore a_1 = 1, \quad (m+1) a_{m+1} + a_{m-1} = 0 \quad (m \geq 1).$$

$$y(0) = 1 \Rightarrow a_0 = 1.$$

$$2a_2 + a_0 = 0, \quad 2a_2 = -1, \quad a_2 = -\frac{1}{2},$$

$$3a_3 + a_1 = 0, \quad 3a_3 = -1, \quad a_3 = -\frac{1}{3}.$$

$$y = \sum_{m=0}^{\infty} a_m x^m = 1 + x - \frac{1}{2} x^2 - \frac{1}{2} x^3 + \cdots.$$

$$= -\frac{e^x}{2}\left[x^2+2x+1-2x-2+2\right] = -\frac{e^x}{2}\left(x^2+1\right).$$

$$y_p = u_1y_1+u_2y_2 = \frac{1}{2}e^x(x+1) - \frac{1}{2}e^x\frac{x^2+1}{x+1} = \frac{e^x}{2}\left(x+1-\frac{x^2+1}{x+1}\right)$$

$$= \frac{e^x}{2}\cdot\frac{(x+1)^2-x^2+1}{x+1} = \frac{xe^x}{x+1}.$$

$$y = c_1y_1+c_2y_2+y_p = c_1(x+1)+\frac{c_2}{x+1}+\frac{xe^x}{x+1}.$$

1b $\quad y(0)=0,\ y'(0)=0.$

$$y' = c_1 - \frac{c_2}{(x+1)^2} + \frac{(x+1)(xe^x+e^x)-xe^x}{(x+1)^2}$$

$$= c_1 - \frac{c_2}{(x+1)^2} + \frac{e^x(x^2+x+1)}{(x+1)^2}.$$

$$y(0) = c_1+c_2 = 0 \Rightarrow c_1 = -c_2,$$

$$y'(0) = c_1-c_2+1 = 2c_1+1 = 0 \therefore c_1 = -\frac{1}{2},\quad c_2 = \frac{1}{2}.$$

$$y = -\frac{x+1}{2}+\frac{1}{2(x+1)}+\frac{xe^x}{x+1}$$

$$= -\frac{(x+1)^2}{2(x+1)}+\frac{1}{2(x+1)}+\frac{2xe^x}{2(x+1)}$$

$$= \frac{-x^2-2x+2xe^x}{2(x+1)} = \frac{x}{2(x+1)}\left(2e^x-x-2\right).$$

MA22S3 Tutorial 10

$$xy'' - xy' + qy = 0, \qquad q \in \mathbb{C}\setminus\{0\}.$$

1 $$y'' - y' + \frac{q}{x} y = 0$$

so there is a singular point at $x=0$ (order 1 — simple pole).

$$\lim_{x\to x_0} (x-x_0)^2 \frac{q}{x} = \lim_{x\to 0} x^2 \frac{q}{x} = 0 < \infty \quad \therefore x_0 \text{ is a regular singular point.}$$

2 Frobenius method about $x_0 = 0$, ansatz is

$$y_1 = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r},$$

$$y_1' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}, \qquad y_1'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2},$$

$$xy_1' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}, \qquad xy_1'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1}$$

$$= \sum_{m=-1}^{\infty} (m+r+1)(m+r) a_{m+1} x^{m+r}, \quad (m \to m+1)$$

$$xy_1'' - xy_1' - qy = \sum_{m=-1}^{\infty} (m+r+1)(m+r) a_{m+1} x^{m+r} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + q \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$= r(r-1) a_0 x^{r-1} + \sum_{m=0}^{\infty} \left[(m+r+1)(m+r) a_{m+1} - (m+r-q) a_m \right] x^{m+r}$$

$$\overset{!}{=} 0 \implies r(r-1) a_0 = 0, \quad r(r-1) = 0 \text{ (indicial equation) } (a_0 \neq 0),$$

$$r_1 = 1, \quad r_2 = 0, \quad r_2 - r_1 \in \mathbb{Z}.$$

Using $r = r_1 = 1$, the recursion relation is

$$(m+2)(m+1) a_{m+1} - (m+1-q) a_m = 0, \quad m \geq 0,$$

$$a_{m+1} = \frac{m+1-q}{(m+2)(m+1)} a_m .$$

So $y_1 = \sum_{m=0}^{\infty} a_m x^{m+1}$ with a_0 free, $a_{m+1} = \frac{m+1-q}{(m+2)(m+1)}$ for $m \geq 0$.

4 2
1 1
20 +30
+
50 +45 +20

3 y_1 is a polynomial if $\exists\, l$ such that $a_m = 0 \;\forall m > l$ (so the series terminates).

$$l = m \implies a_{m+1} = 0 \implies m+1-q = 0, \quad q = m+1$$

so $y_1 \in P_n$ if $q \in \mathbb{Z}^+$, $n = q$. (Trivial solution $y_1 = 0$ implies $a_0 = 0$.)

4 $q = 1$, $y_2 = y_1 \log(x) + \sum_{m=0}^{\infty} A_m x^{m+r_2}$. $r_2 = 0$ and $y_1 \in P_1$ (from 2, 3).

$$a_{m+1} = \frac{m}{(m+1)(m+2)} \implies \{a_i\}_{i=1}^{\infty} \to 0 \quad \therefore y_1 = a_0 x \in P_1.$$

$$y_2 = a_0 x \log(x) + \sum_{m=0}^{\infty} A_m x^m,$$

$$y_2' = a_0 + a_0 \log(x) + \frac{d}{dx}\sum_{m=0}^{\infty} A_m x^m,$$

$$y_2'' = \frac{a_0}{x} + \frac{d^2}{dx^2}\sum_{m=0}^{\infty} A_m x^m,$$

$$xy_2'' - xy_2' + qy_2 = a_0 - a_0 x - a_0 x \log(x) + a_0 x \log(x) + \left(x\frac{d^2}{dx^2} - x\frac{d}{dx} + 1\right)\sum_{m=0}^{\infty} A_m x^m$$

$$= a_0 - a_0 x + \left(x\frac{d^2}{dx^2} - x\frac{d}{dx} + 1\right)\sum_{m=0}^{\infty} A_m x^m.$$

Alternative: use result from 2 with $r=0$, $q=1$, $A_m = a_m$.

$$x\frac{d}{dx}\sum_{m=0}^{\infty} A_m x^m = \sum_{m=0}^{\infty} m A_m x^m,$$

$$x\frac{d^2}{dx^2}\sum_{m=0}^{\infty} A_m x^m = \sum_{m=0}^{\infty} m(m-1) A_m x^{m-1}$$

$$= \sum_{m=-1}^{\infty} m(m+1) A_{m+1} x^m = \sum_{m=0}^{\infty} m(m+1) A_{m+1} x^m.$$

$$xy_2'' - xy_2' + qy_2 = a_0 - a_0 x + \sum_{m=0}^{\infty}\left[m(m+1)A_{m+1} - mA_m + A_m\right]x^m \overset{!}{=} 0,$$

$$m(m+1)A_{m+1} - (m-1)A_m = 0, \quad m \geq 2,$$

$$A_{m+1} = \frac{m-1}{m(m+1)} A_m, \quad m \geq 2.$$

For $m=0$, $a_0 + A_0 = 0$. For $m=1$, $2A_2 - a_0 = 0$. $A_0 = -a_0$, $A_2 = \frac{a_0}{2}$, and

$$y_2 = a_0 x \log(x) + \sum_{m=0}^{\infty} A_m x^m, \quad A_0, A_1 \text{ free.}$$

5. $y = c_1 y_1 + c_2 y_2$, but $c_1 y_1 = c_1 a_0 x$. This term already exists in y_2 as $A_1 x$, so the two are equivalent up to $A_1 \to A_1 + c_1 a_0$, hence y_2 is the most general form for $q = 1$.

Alternatively, if we had chosen values for the undetermined coefficients $y = c_1 y_1 + c_2 y_2$ would be the general solution, c_1, c_2 undetermined.