Fourier Coefficients

Fionn Fitzmaurice fionn@maths.tcd.ie

We will define the inner product of (real-valued) functions f, $g \in L^2[-\frac{L}{2}, \frac{L}{2}]$ as

$$\langle f|g\rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)g(t)$$

for some $L \in \mathbb{R}$.

Define

$$c_n : t \mapsto \cos\left(\frac{2\pi nt}{L}\right), \qquad s_n : t \mapsto \sin\left(\frac{2\pi nt}{L}\right).$$

Notice that $c_0(t) = 1$ and $s_0(t) = 0$ for all $t \in \mathbb{R}$.

For positive m, n,

$$\langle c_m|c_n\rangle = \frac{L}{2}\delta_{mn}, \qquad \langle s_m|s_n\rangle = \frac{L}{2}\delta_{mn}, \qquad \langle c_m|s_n\rangle = 0,$$

so these functions are orthogonal. In fact, they span L^2 and therefore form a basis.

We would like to be able to express a periodic function $f \in L^2$ as a linear combination of an infinite number of basis vectors $\{c_n, s_n\}_{n=0}^{\infty}$. This is the Fourier series.

Recall that the projection $P_w(v)$ of v onto w is given by

$$P_w(v) = \frac{\langle v|w\rangle}{\langle w|w\rangle} w.$$

We can use this to determine the coefficients of the Fourier series, as

$$\begin{aligned}\frac{\langle f|c_n\rangle}{\langle c_n|c_n\rangle} &= \frac{2}{L}\langle f|c_n\rangle \\ &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)\cos\left(\frac{2\pi nt}{L}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\langle f|s_n\rangle}{\langle s_n|s_n\rangle} &= \frac{2}{L}\langle f|s_n\rangle \\ &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)\sin\left(\frac{2\pi nt}{L}\right).\end{aligned}$$

Using this we get the Euler formulae

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)\cos\left(\tfrac{2\pi nt}{L}\right), \qquad b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)\sin\left(\tfrac{2\pi nt}{L}\right)$$

which define a_n and b_n for all non-negative n. Notice that this gives us

$$b_0 = 0, \qquad a_0 = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t).$$

Let $f \in L^2$ and $f(t+L) = f(t)$ for all $t \in \mathbb{R}$. We might now be tempted to write down a naïve expression for the Fourier series of f as

$$f(t) = \sum_{n=0}^{\infty}\left[\frac{\langle f|c_n\rangle}{\langle c_n|c_n\rangle}c_n + \frac{\langle f|s_n\rangle}{\langle s_n|s_n\rangle}s_n\right] = \sum_{n=0}^{\infty}\left[a_n\cos\left(\tfrac{2\pi nt}{L}\right) + b_n\sin\left(\tfrac{2\pi nt}{L}\right)\right].$$

We expect $\langle f|c_n\rangle = \frac{L}{2}a_n$ and $\langle f|s_n\rangle = \frac{L}{2}b_n$. However $\langle f|c_0\rangle \neq \frac{L}{2}a_0$ as

$$\begin{aligned}\langle f|c_0\rangle &= a_0\langle c_0|c_0\rangle + a_1\langle c_1|c_0\rangle + \cdots \\ &\quad + b_0\langle s_0|c_0\rangle + b_1\langle s_1|c_0\rangle + \cdots \\ &= a_0\langle c_0|c_0\rangle \\ &= a_0\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \\ &= a_0 L.\end{aligned}$$

This is because our definition of a_0 from the Euler formulae doesn't follow from the derivation since $\langle c_n|c_n\rangle = L \neq \frac{L}{2}$ when $n = 0$.

One solution to this is to redefine a_0 such that it is inconsistent with the general formula for a_n but consistent with our expression for the Fourier series of f above. The conventional approach is to instead redefine our series to be

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\tfrac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\tfrac{2\pi nt}{L}\right).$$

Then

$$\begin{aligned}\langle f|c_0\rangle &= \frac{a_0}{2}\langle c_0|c_0\rangle \\ &= \frac{L}{2}a_0 \\ &= \frac{L}{2}\cdot\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t) \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, f(t)\end{aligned}$$

as expected.