- $$\langle f(x) \,|\, g(x) \rangle = \int_{-a}^{a} f(x)\, g(x)\, dx.$$

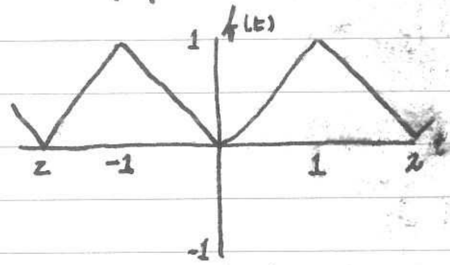
Let f be any odd function, g any even function. Then

$$f(-x)\,g(-x) = -f(x)\,g(x).$$

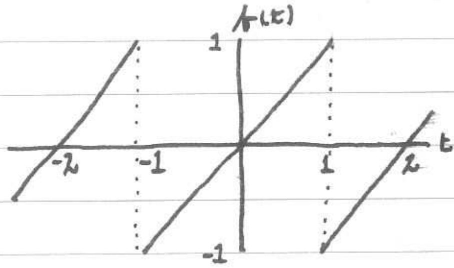
$$\int_{-a}^{a} f(x)\, g(x)\, dx = \int_{-a}^{0} dx\, f(x)\, g(x) + \int_{0}^{a} dx\, f(x)\, g(x)$$
$$= \int_{+a}^{0} (-dy)\, f(-y)\, g(-y) + \int_{0}^{a} dx\, f(x)\, g(x)$$
$$= \int_{0}^{a} dy\, f(-y)\, g(-y) + \int_{0}^{a} dx\, f(x)\, g(x)$$
$$= -\int_{0}^{a} dy\, f(y)\, g(y) + \int_{0}^{a} dx\, f(x)\, g(x) = 0.$$

- $f(t) = t$, $0 \le t < 1$ extended to $-2 < t < 2$ of period 2.

• even $\Rightarrow f(-t) = f(t)$, so



• odd, $\Rightarrow f(-t) = -f(t)$, so



- $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{L}\right)$$
$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{L}\right) \qquad (\text{since } f(t) \text{ is odd, } a_n = 0).$$

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t) \sin\left(\frac{2\pi n t}{L}\right) dt$$
$$= \int_{-1}^{1} t \sin(\pi n t)\, dt$$
$$= 2\int_{0}^{1} \underbrace{t}_{u}\, \underbrace{\sin(\pi n t)\, dt}_{dv}$$

$$= 2\int_0^1 t \sin(\pi n t)\, dt \qquad v = \int dv = \int \sin(\pi n t)\, dt$$

$$= 2\left[-\frac{t\cos(\pi n t)}{\pi n}\right]_0^1 + 2\cdot\frac{1}{\pi n}\int_0^1 \cos(\pi n t)\, dt \qquad = -\frac{1}{\pi n}\cos(\pi n t)$$

$$= 2\cdot -\frac{\cos(\pi n)}{\pi n} + \frac{2}{\pi n}\Big[\sin(\pi n t)\Big]_0^1 \cdot \frac{1}{\pi n}$$

$$= -\frac{2\cos(\pi n)}{\pi n} + \frac{2}{(\pi n)^2}\cdot 0$$

$$= -\frac{2}{\pi n}\cos(\pi n)$$

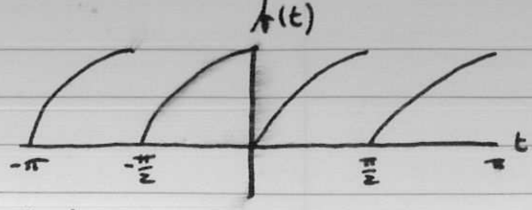
$$= -\frac{2}{\pi n}(-1)^n,$$

So

$$f(t) = \sum_{n=1}^{\infty} -\frac{2}{\pi n}(-1)^n \sin(\pi n t)$$

$$= \frac{2}{\pi}\left(\sin(\pi t) - \frac{1}{2}\sin(2\pi t) + \frac{1}{3}\sin(3\pi t) - \cdots\right).$$

$$f(t) = \sin(t), \quad 0 < t < \frac{\pi}{2},$$



$$f\left(t + \frac{\pi}{2}\right) = f(t) \;\; \forall t,$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{L}}, \quad c_n = \frac{1}{L}\int_{t_0}^{t_0+L} f(t) e^{-\frac{2\pi i n t}{L}} dt$$

$$c_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(t) e^{-4int} dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{e^{it} - e^{-it}}{2i} e^{-4int} dt$$

$$= \frac{1}{i\pi}\int_0^{\frac{\pi}{2}} \left(e^{it-4int} - e^{-it-4int}\right) dt$$

$$= \frac{1}{i\pi}\int_0^{\frac{\pi}{2}} \left(e^{(i-4in)t} - e^{(-i-4in)t}\right) dt$$

$$= \frac{1}{i\pi}\left[\frac{e^{(i-4in)t}}{i(1-4n)} + \frac{e^{(-i-4in)t}}{i(1+4n)}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{i\pi}\left[\frac{e^{it}e^{-4int}}{i(1-4n)} + \frac{e^{-it}e^{-4int}}{i(1+4n)}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{\pi}\left[\frac{e^{it}e^{-4int}}{1-4n} + \frac{e^{-it}e^{-4int}}{1+4n}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{\pi}\left(\frac{1}{1-4n} e^{i\frac{\pi}{2}} e^{-2\pi i n} + \frac{1}{1+4n} e^{-i\frac{\pi}{2}} e^{-2\pi i n} - \frac{1}{1-4n} - \frac{1}{1+4n}\right)$$

$$= -\frac{1}{\pi}\left(\frac{1}{1-4n}\left(e^{i\frac{\pi}{2}} - 1\right) + \frac{1}{1+4n}\left(e^{-i\frac{\pi}{2}} - 1\right)\right)$$

$$= -\frac{1}{\pi}\left(\frac{1}{1-4n}(i-1) + \frac{1}{1+4n}(-i-1)\right)$$

$$= -\frac{1}{\pi} \cdot \frac{(i-1)(1+4n) + (-i-1)(1-4n)}{(1-4n)(1+4n)}$$

$$= -\frac{1}{\pi} \frac{8in - 2}{1 - 16n^2}$$

$$= \frac{2}{\pi} \cdot \frac{4in - 1}{16n^2 - 1}.$$

$$f(t) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{4in-1}{16n^2-1} \cdot e^{4int}.$$

At $t=0$, the Fourier series is $\frac{1}{2}$.

$$\frac{1}{2} = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4in-1}{16n^2-1}$$

$$= \frac{2}{\pi}\left[\sum_{n=-1}^{-\infty}\frac{4in-1}{16n^2-1} + 1 + \sum_{n=1}^{\infty}\frac{4in-1}{16n^2-1}\right]$$

$$= \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{-4in-1}{16n^2-1} + 1 + \sum_{n=1}^{\infty}\frac{4in-1}{16n^2-1}\right]$$

$$= \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty}\left(\frac{-4in-1+4in-1}{16n^2-1}\right)\right]$$

$$= \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{-2}{16n^2-1}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{16n^2-1},$$

$$\sum_{n=1}^{\infty}\frac{1}{16n^2-1} = -\frac{\pi}{4}\left(\frac{1}{2}-\frac{2}{\pi}\right)$$

$$= \frac{1}{2} - \frac{\pi}{8}.$$

$$= \frac{4-\pi}{8}.$$

Parseval's theorem:

$$\frac{1}{L}\int_{t_0}^{t_0+L} f(t)^2\,dt = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

- $f(t) = t$, $0 \leq t < 1$, $L = 2$, odd extension. $a_n = 0$, $b_n = -\frac{2}{\pi n}(-1)^n$.

$$\frac{1}{L}\int_{t_0}^{t_0+L} f(t)^2\,dt = \frac{1}{2}\int_{-1}^{1} t^2\,dt$$
$$= \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{3}.$$

$$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{2}{\pi n}(-1)^n\right)^2$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

$$\frac{1}{3} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

- $\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$, $\tilde{f}(-\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$

Let $u = -t$,

$$\tilde{f}(-\omega) = \frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty} -f(-u)e^{-i\omega u}\,du$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-u)e^{-i\omega u}\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt,$$

so the Fourier transformation of $f(-t)$ is $\tilde{f}(-\omega)$. If f is even,

$$\tilde{f}(-\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \tilde{f}(\omega)$$

and thus $\tilde{f}$ is even. If f is odd,

$$\tilde{f}(-\omega) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = -\tilde{f}(\omega)$$

and $\tilde{f}$ is odd too. This means the Fourier transformation preserves parity.

- $$\int_{-\infty}^{\infty}[\delta(t-1)+\delta(t+1)]\,e^{t}\,dt = e + e^{-1}$$

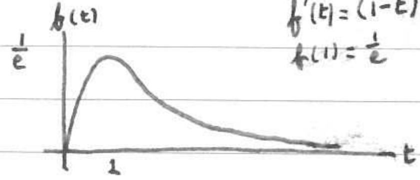
$$\int_{-2}^{2}\delta(t-3)(t^2-4t)\,dt = 0$$

$$\int_{-2}^{2}\delta(2t)(t+3)\,dt = \frac{1}{2}\cdot 3$$

$-f'(0)$

$$\int_{-2}^{2}\delta'(t)\,4t\,dt = \Big[4t\,\delta(t)\Big]_{-2}^{2} - \int_{-2}^{2}4\,\delta(t)\,dt = -4$$

- $$f(t) = \begin{cases} te^{-t}, & t>0 \\ 0, & t\le 0 \end{cases}$$



$f'(t) = (1-t)e^{-t} = 0 \Rightarrow t = 1,$
$f(1) = \frac{1}{e}$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt$$

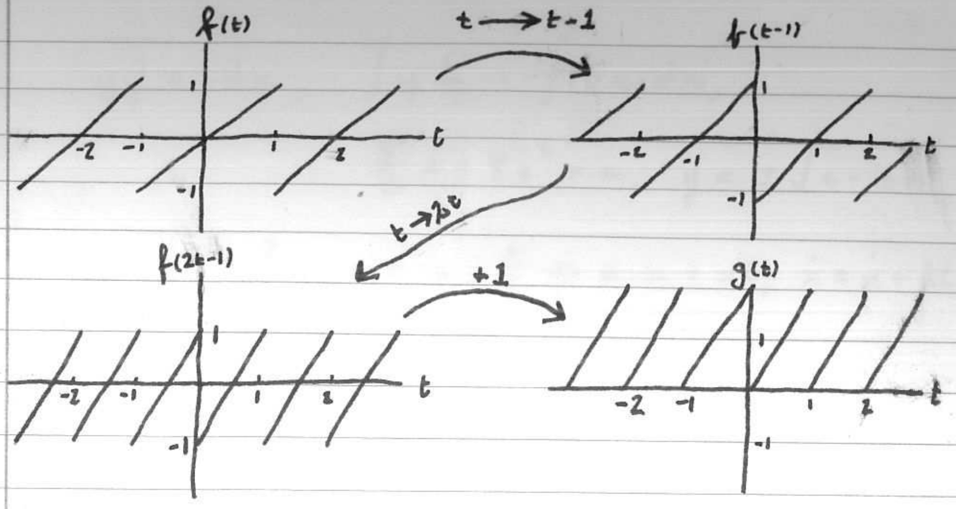
$$= \frac{1}{\sqrt{2\pi}}\int_{\cancel{-\infty}\,0}^{\infty} te^{-t}e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\cancel{-\infty}\,0}^{\infty} t\,e^{-(1+i\omega)t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left(\left[\frac{t}{-(1+i\omega)}e^{-(1+i\omega)t}\right]_0^{\infty} - \int_0^{\infty}\frac{1}{-(1+i\omega)}e^{-(1+i\omega)t}\,dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{1}{1+i\omega}e^{-(1+i\omega)t}\,dt$$

$$= -\frac{1}{\sqrt{2\pi}}\,\frac{1}{(1+i\omega)^2}\cdot e^{-(1+i\omega)}\Big|_0^{\infty} = \frac{+1}{\sqrt{2\pi}}\cdot\frac{1}{(1+i\omega)^2}.$$



$$f(t) = t\,,\ |t|<1\,,\ f(t+2) = f(t)\,,\quad g(t) = 2t\,,\ 0<t<1\,,\ g(t+1) = g(t).$$

$$\begin{aligned}
g(t) &= f(2t-1) + 1\,, \\
&= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(2\pi n t - \pi n) \\
&= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\Big(\sin(2\pi n t)\cos(\pi n) - \cos(2\pi n t)\sin(\pi n)\Big) \\
&= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\Big(\sin(2\pi n t)(-1)^n - 0\Big) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2\pi n t).
\end{aligned}$$

$$h(t) = \begin{cases} 1 & \text{if } |t|<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}\,,\quad (h*h)(t) = \begin{cases} 1-t & \text{if } |t|<1 \\ 0 & \text{otherwise} \end{cases}\,,\quad \tilde{h}(\omega) = \sqrt{\frac{2}{\omega}}\,\frac{\sin(\frac{\omega}{2})}{\omega}.$$

$$\int_{-\infty}^{\infty}|h(t)|^2\,dt = \int_{-\infty}^{\infty}|\tilde{h}(\omega)|^2\,d\omega \qquad \left(x = \frac{\omega}{2}\right)$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}dt = \frac{2}{\pi}\int_{-\infty}^{\infty}d\omega\,\frac{\sin^2(\frac{\omega}{2})}{\omega^2} = \frac{2}{\pi}\int_{-\infty}^{\infty}2\,dx\,\frac{\sin^2(x)}{4x^2} = \frac{1}{\pi}\int_{-\infty}^{\infty}dx\,\frac{\sin^2(x)}{x^2}$$

$$\therefore\ \pi = \int_{-\infty}^{\infty}dx\,\frac{\sin^2(x)}{x^2}.$$

$$\mathcal{F}[(h*h)(t)] = \sqrt{2\pi}\,\frac{2}{\pi}\left(\frac{\sin(\frac{\omega}{2})}{\omega}\right)^2 = \frac{2\sqrt{2}}{\sqrt{\pi}}\,\frac{\sin^2(\frac{\omega}{2})}{\omega^2}\,,$$

$$\int_{-\infty}^{\infty}dt\,|(h*h)(t)|^2 = \frac{8}{\pi}\int_{-\infty}^{\infty}d\omega\,\frac{\sin^4(\frac{\omega}{2})}{\omega^4} = \frac{8}{\pi}\int_{-\infty}^{\infty}2\,dx\,\frac{\sin^4(x)}{16x^4} = \frac{1}{\pi}\int_{-\infty}^{\infty}dx\,\frac{\sin^4(x)}{x^4}\,,$$

$$\int_{-\infty}^{\infty}dt\,|(h*h)(t)|^2 = \int_{-1}^{1}dt\,(1-|t|)^2 = 2\int_{0}^{1}dt\,(1-t)^2 = 2\int_{0}^{1}dt\,(1-2t+t^2) = 2\left[t - t^2 + \frac{t^3}{3}\right]_0^1 = \frac{2}{3}\,,$$

$$\therefore\ \int_{-\infty}^{\infty}dx\,\frac{\sin^4(x)}{x^4} = \frac{2\pi}{3}.$$

a $\quad yy' = -16x, \quad \int y\,dy = -\int 16x\,dx,$

$$\frac{y^2}{2} = -8x^2 + c, \quad y = \pm\sqrt{c - 16x^2}$$

b $\quad \dot{x} = \frac{t^2}{x^2} + \frac{x}{t}, \quad$ let $u = \frac{x}{t} \Rightarrow x = ut, \; \dot{x} = u + \dot{u}t.$

$$u + \dot{u}t = \frac{1}{u^2} + u, \quad \dot{u}t = \frac{1}{u^2},$$

$$\dot{u}u^2 = \frac{1}{t},$$

$$\int u^2\,du = \int \frac{dt}{t},$$

$$\frac{u^3}{3} = \log(|t|) + c, \quad u = (3\log(|t|) + c)^{\frac{1}{3}},$$

$$x = ut = t(3\log(|t|) + c)^{\frac{1}{3}}.$$

c $\quad \dot{x} - \log(t)x = t^t, \quad I(t) = -\int dt \log(t) = -t\log(t) + t$

$$e^{I(t)}(\dot{x} - \log(t)x) = e^{I(t)} e^{t\log(t)}$$

$$e^{t - t\log(t)}\dot{x} - e^{t - t\log(t)}\log(t)x = e^t$$

$$\frac{d}{dt}\left(e^{t - t\log(t)}x\right) = e^t \Rightarrow e^{t - t\log(t)}x = e^t + c,$$

$$x = e^{t\log(t)} + ce^{t\log(t) - t}$$

$$= t^t(1 + ce^{-t}).$$

3 $\quad \dot{y} + (y - u)\sin(t) = 0, \quad y(\pi) = 2u.$

$$\dot{y} + y\sin(t) = u\sin(t), \quad I(t) = \int \sin(t)\,dt = -\cos(t)$$

$$e^{-\cos(t)}\dot{y} + e^{-\cos(t)}y\sin(t) = e^{-\cos(t)}u\sin(t)$$

$$\frac{d}{dt}\left(e^{-\cos(t)}y\right) = ue^{-\cos(t)}\sin(t)$$

$$e^{-\cos(t)}y = ue^{-\cos(t)} + c$$

$$y = u + ce^{\cos(t)}, \quad y(\pi) = u + ce^{-1} = 2u, \quad c = ue, \quad y = u(1 + e^{1+\cos(t)}).$$

- $\ddot{f}(t) + 4\dot{f}(t) + 5f(t) = 0$, $f(0) = 2$, $\dot{f}(0) = -5$.

$f(t) = e^{\lambda t} \Rightarrow \lambda^2 + 4\lambda + 5 = 0$, $\lambda = -2 \pm i$,

$f(t) = e^{-2t}(c_1 \sin(t) + c_2 \cos(t))$,

$f'(t) = -2c_1 e^{-2t} \sin(t) + e^{-2t} c_1 \cos - 2c_2 e^{-2t} \cos(t) - c_2 e^{-2t} \sin(t)$

$= e^{-2t}\big((c_1 - 2c_2)\cos(t) - (2c_1 + c_2)\sin(t)\big)$.

$f(0) = 2 \Rightarrow c_2 = 2$, $\dot{f}(0) = -5 \Rightarrow c_1 = -1$.

- $\ddot{r}(t) - 4\dot{r}(t) + 4r(t) = 0$, $r(2) = 4$, $\dot{r}(2) = 2$.

$r = e^{\lambda t} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$, $\lambda = 2$,

$r(t) = (c_1 + tc_2)e^{2t}$, $\dot{r}(t) = (2c_1 + c_2 + 2tc_2)e^{2t}$,

$r(2) = 4$, $(c_1 + 2c_2)e^4 = 4$, $\dot{r}(2) = 2$, $(2c_1 + c_2 + 4c_2)e^4 = 2$.

$c_1 + 2c_2 = 4e^{-4}$, $2c_1 + 5c_2 = 2e^{-4}$,

$2(4e^{-4} - c_2) + 5c_2 = 2e^{-4} \Rightarrow c_2 = -6e^{-4}$, $c_1 = 16e^{-4}$,

$r(t) = (16 - 6t)e^{2t-4}$

- $h''(x) - 4h'(x) - 3h(x) = 0$, $h(0) = 12$, $h'(0) = 0$.

$h(x) = e^{\lambda x} \Rightarrow \lambda^2 - 4\lambda - 3 = 0$, $\lambda_\pm = 2 \pm \sqrt{7}$.

$h(x) = c_1 e^{(2+\sqrt{7})x} + c_2 e^{(2-\sqrt{7})x}$, $h'(x) = (2+\sqrt{7})c_1 e^{(2+\sqrt{7})x} + (2-\sqrt{7})c_2 e^{(2-\sqrt{7})x}$.

$h'(0) = 12 \Rightarrow c_1 + c_2 = 12$, $h'(0) = 0 \Rightarrow (2+\sqrt{7})c_1 + (2-\sqrt{7})c_2 = 0$

$c_1 = 12 - c_2$, $(2+\sqrt{7})(12 - c_2) + (2-\sqrt{7})c_2 = 0$,

$24 + 12\sqrt{7} \cancel{-2c_2} - \sqrt{7}c_2 \cancel{+2c_2} - \sqrt{7}c_2 = 0$

$24 + 12\sqrt{7} = 2\sqrt{7}c_2$, $c_2 = 6 + \frac{12}{\sqrt{7}}$, $c_1 = 6 - \frac{12}{\sqrt{7}}$.

$h(x) = \left(6 - \frac{12}{\sqrt{7}}\right)e^{(2+\sqrt{7})x} + \left(6 + \frac{12}{\sqrt{7}}\right)e^{(2-\sqrt{7})x}$

$$x^2y'' + 2xy' + 6y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

Ansatz: $y = x^m$. $\qquad \left(am(m-1) + bm + c = 0, \quad m^2 - m + 2m + 6 = 0 \right)$

$$m^2 + m + 6 = 0, \quad m_\pm = \frac{-1 \pm i\sqrt{23}}{2}$$

$$y = c_1 x^{m_+} + c_2 x^{m_-}$$

$$= c_1 x^{-\frac{1}{2} + \frac{i\sqrt{23}}{2}} + c_2 x^{-\frac{1}{2} - \frac{i\sqrt{23}}{2}}$$

$$= c_1 x^{-\frac{1}{2}} e^{\frac{i\sqrt{23}}{2}\log(x)} + c_2 x^{-\frac{1}{2}} e^{-\frac{i\sqrt{23}}{2}\log(x)}$$

$$= c_1 x^{-\frac{1}{2}} \left(\cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + i\sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) \right)$$

$$+ c_2 x^{-\frac{1}{2}} \left(\cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) - i\sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) \right)$$

$$= x^{-\frac{1}{2}} \left[c_1\cos(\quad) + c_2\cos(\quad) + c_1 i\sin(\quad) - c_2 i \sin(\quad) \right]$$

$$= x^{-\frac{1}{2}} \left[c_3 \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + c_4 i \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) \right] \qquad \begin{aligned} c_3 &= c_1 + c_2 \\ c_4 &= c_1 - c_2 \\ c_5 &= -i(c_1 - c_2) \end{aligned}$$

$$= x^{-\frac{1}{2}} c_5 \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + c_3 x^{-\frac{1}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right)$$

$$= c_1 x^{-\frac{1}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + c_2 x^{-\frac{1}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right). \qquad \text{(renaming)}.$$

$$y' = -\tfrac{1}{2} c_1 x^{-\frac{3}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + c_1 x^{-\frac{1}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) \cdot \tfrac{\sqrt{23}}{2} \cdot \tfrac{1}{x}$$

$$- \tfrac{1}{2} c_2 x^{-\frac{3}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) - c_2 x^{-\frac{1}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) \cdot \tfrac{\sqrt{23}}{2} \cdot \tfrac{1}{x}$$

$$= -\tfrac{1}{2} c_1 x^{-\frac{3}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) - c_2 \cdot \tfrac{\sqrt{23}}{2} x^{-\frac{3}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right)$$

$$- \tfrac{1}{2} c_2 x^{-\frac{3}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + c_1 x^{-\frac{3}{2}} \cdot \tfrac{\sqrt{23}}{2} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right)$$

$$= -\tfrac{1}{2}(c_1 + \sqrt{23}c_2) x^{-\frac{3}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + \tfrac{1}{2}(\sqrt{23}c_1 - c_2) x^{-\frac{3}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right).$$

$y(1) = 1 \Rightarrow c_1 \sin(0) + c_2\cos(0) = 1 \Rightarrow c_2 = 1$, $y'(1) = 0 \Rightarrow$

$$\cancel{-\tfrac{1}{2}(1 + \sqrt{23}c_2)\sin(0) + \tfrac{1}{2}(\sqrt{23}c_1 - c_2)\cos(0) = 0}$$

$$\tfrac{1}{2}(\sqrt{23}c_1 - 1) = 0$$

$$c_1 = \tfrac{1}{\sqrt{23}}$$

$$y = \frac{1}{\sqrt{23}} x^{-\frac{1}{2}} \sin\left(\tfrac{\sqrt{23}}{2}\log(x)\right) + x^{-\frac{1}{2}} \cos\left(\tfrac{\sqrt{23}}{2}\log(x)\right).$$

- $x^2 f'' - 2x f' + 2f = 6x^4 + 2\log^2(x) - 5, \quad x > 0.$

Homogeneous equation $x^2 f'' - 2x f' + 2f = 0$, ansatz ~~m~~ $f = x^m$ (Euler).

$$m(m-1) - 2m + 2 = 0 \Rightarrow m_1 = 1, \; m_2 = 2, \; f_1 = x, \; f_2 = x^2.$$

$$f_p = ax^4 + b\log^2(x) + c\log(x) + d,$$

$$f_p' = 4ax^3 + \frac{2b}{x}\log(x) + \frac{c}{x}, \quad f_p'' = 12ax^2 - \frac{2b}{x^2}\log(x) + \frac{2b}{x^2} - \frac{c}{x^2}.$$

$$\cancel{x^2\left(12ax^2 - \frac{2b}{x^2}(\log(x)+1) - \frac{c}{x^2}\right) - 2x(ax^4 + b\log^2(x) + c\log(x))}$$

$$12ax^4 - 2b\log(x) + 2b - c - 8ax^4 - 4b\log(x) - 2c + 2ax^4 + 2b\log^2(x) + 2c\log(x) + 2d = 6x^4 + 2\log^2(x) - 5.$$

$$6ax^4 + 2b\log^2(x) + (-6b + 2c)\log(x) + 2b - 3c + 2d = 6x^4 + 2\log^2(x) - 5,$$

$$a = 1, \; b = 1, \; c = 3, \; d = 1, \; f_p = x^4 + \log^2(x) + 3\log(x) + 1,$$

$$f = f_c + f_p = c_1 x + c_2 x^2 + x^4 + \log^2(x) + 3\log(x) + 1.$$

- $f'' - 2f' + f = \frac{e^x}{x^2}, \quad x > 0.$

For the homogeneous case, ansatz $f = e^{\lambda x}$, $\lambda^2 - 2\lambda + 1 = 0$, $\lambda_\pm = 1$, $f_1 = e^x$, $f_2 = xe^x$.

$$f_p = u_1 f_1 + u_2 f_2 = u_1 e^x + u_2 x e^x,$$

$$f_p' = u_1 e^x + u_2 e^x + u_2 x e^x + u_1' e^x + u_2' x e^x.$$

$$u_1 = -\int \frac{g(x) f_2(x)}{W(f_1,f_2)} dx, \quad u_2 = \int \frac{g(x) f_1(x)}{W(f_1,f_2)} dx, \quad W(f_1,f_2) = e^{2x}$$

Impose $u_1' e^x + u_2' x e^x = 0$ (to avoid $u_{1,2}''$).

$$f_p'' = \underline{u_1} e^x + 2u_2 e^x + \underline{u_2} x e^x + u_1' e^x + u_2' e^x + u_2' x e^x.$$

$$f_p'' - 2\cancel{u_1} e^x - 2\cancel{u_2} e^x - \cancel{u_2} x e^x - u_1' e^x - u_2' x e^x + \cancel{2u_1 e^x} + \cancel{2u_2} x e^x = \frac{e^x}{x^2}$$

$$\frac{e^x}{x^2} = u_1' e^x + u_2'(e^x + xe^x) \Rightarrow u_2' = \frac{1}{x^2}, \; u_1' = -\frac{1}{x}, \quad u_1 = -\log(x), \; u_2 = -\frac{1}{x}.$$

$$f_p = -e^x\log(x) - e^x, \quad f = c_1 e^x + c_2 x e^x - e^x\log(x) \; (-e^x).$$

$y'' - 2xy' + 2y = 0.$

$y = \sum_{m=0}^{\infty} a_m x^m$, $\quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$, $\quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

$= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$,

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - 2\sum_{m=1}^{\infty} m a_m x^m + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} x^m \left[(m+2)(m+1) a_{m+2} - 2m a_m + 2a_m \right] = 0$$

Let $m=0$, then $2a_{m+2} + 2a_m = 0$, $2a_2 + 2a_0 = 0$. For $m \geq 1$,

$$(m+2)(m+1) a_{m+2} - 2(m-1) a_m = 0,$$

$$a_{m+2} = \frac{2(m-1)}{(m+2)(m+1)} a_m.$$

If $\exists n$ such that $a_m = 0$ $\forall m > n$ then there exists a polynomial solution. When $m=1$, $a_3 = 0$. Then $a_{2k+1} = 0$ $\forall k \in \mathbb{N}^+$, so if $a_0 = 0$ $y = a_1 x$ is a polynomial solution.

$(x^2+2)y'' - xy' + 4y = 0.$

$m \to m+2$ $\quad \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^m + 2\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=1}^{\infty} m a_m x^m + 4\sum_{m=0}^{\infty} a_m x^m = 0,$$

$$\sum_{m=0}^{\infty} x^m \left[m(m-1) a_m + 2(m+1)(m+2) a_{m+2} - m a_m + 4a_m \right] = 0,$$

$$\sum_{m=0}^{\infty} x^m \left[2(m+2)(m+1) a_{m+2} + (m^2 - 2m + 4) a_m \right] = 0,$$

$$a_{m+2} = -\frac{m^2 - 2m + 4}{2(m+2)(m+1)} a_m.$$

If $y(0) = 2$, $y'(0) = 1$, then $a_0 = 2$ and $a_1 = 1$.

$a_2 = -a_0 = -2$, $\quad a_3 = -\frac{1}{4} a_1 = -\frac{1}{4}$, $\quad a_4 = -\frac{1}{6} a_2 = \frac{1}{3}$, etc.

$$y = 2 + x - 2x^2 - \frac{1}{4}x^3 + \frac{1}{3}x^4 + \cdots$$