Plancharel's Theorem

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Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a complex-valued function. We will use the convention that

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{-ikx}, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k\, \tilde{f}(k) e^{ikx}.$$

We want to prove that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, |f(x)|^2 = \int_{-\infty}^{\infty} \mathrm{d}k\, |\tilde{f}(k)|^2$$

or, equivalently, $\|f\|_2^2 = \|\tilde{f}\|_2^2$. Under a different convention for the Fourier transformation, there might be some factors of 2π floating around.

$$\begin{aligned}
\|f\|_2^2 = \int_{-\infty}^{\infty} \mathrm{d}x\, |f(x)|^2 &= \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)^* f(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \left[\int_{-\infty}^{\infty} \mathrm{d}k \tilde{f}(k)^* e^{-ikx} \cdot \int_{-\infty}^{\infty} \mathrm{d}k'\, \tilde{f}(k') e^{ik'x} \right] \\
&= \frac{1}{2\pi} \iiint_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}k\, \mathrm{d}k'\, \tilde{f}(k)^* \tilde{f}(k') e^{ik'x} e^{-ikx} \\
&= \frac{1}{2\pi} \iiint_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}k\, \mathrm{d}k'\, \tilde{f}(k)^* \tilde{f}(k') e^{ix(k'-k)},
\end{aligned}$$

where $z^* = \bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The Fourier transformation $\tilde{\delta}(k)$ of $\delta(x)$ is

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x) e^{-ikx} = \frac{1}{\sqrt{2\pi}},$$

so

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx}.$$

Using this,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x\, |f(x)|^2 &= \iint_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{d}k'\, \delta(k' - k) \tilde{f}(k)^* \tilde{f}(k') \\
&= \int_{-\infty}^{\infty} \mathrm{d}k\, \tilde{f}(k)^* \tilde{f}(k) \\
&= \int_{-\infty}^{\infty} \mathrm{d}k\, |\tilde{f}(k)|^2 = \|\tilde{f}\|_2^2,
\end{aligned}$$

giving $\|f\|_2^2 = \|\tilde{f}\|_2^2$. □