The Derivative of the Fourier Transformation

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Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a complex-valued function. We will use the convention that

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, f(x) e^{-ikx}, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \, \tilde{f}(k) e^{ikx}$$

and the slightly clunky notation of $\mathcal{F}(f(x)) = \tilde{f}(k)$ to denote the Fourier transformation.

We would like to find an expression for the Fourier transformation of $f'(x)$. The brutish way is to integrate by parts like

$$\begin{aligned}
\mathcal{F}(f'(x)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, f'(x) e^{-ikx} \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[f(x) e^{-ikx} \right]_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} \mathrm{d}x \, f(x) e^{-ikx} \right) \\
&= ik \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, f(x) e^{-ikx} \\
&= ik \tilde{f}(k),
\end{aligned}$$

where the boundary term vanishes since

$$\lim_{|x| \to \infty} f(x) = 0.$$

It's very easy to justify this limit physically; mathematically I think this works since the existence of the Fourier transformation of f and f' presumes the integrability of f and f'. If f is unbounded, $\tilde{f}$ will be discontinuous. There's a myth that $f(x) \to 0$ as $x \to \infty$ because $f \in L^2$, which probably comes from this. Again, I think the only functions in L^2 which won't satisfy this will be suitably pathological.

It is simpler to notice that

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} \mathrm{d}k \, \tilde{f}(k) e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \, ik \tilde{f}(k) e^{ikx},
\end{aligned}$$

where in the last line we make use of the Leibniz integral rule, and we are compelled to identify $\mathcal{F}(f'(x)) = ik\tilde{f}(k)$.

As an aside, the momentum operator $P = -i\hbar \frac{\partial}{\partial x}$ has eigenvalues p which represent observable momenta. Given the above result and the fact that $p = \hbar k$, you should be able to see why (or at least that it is consistent).