THE STANDARD MODEL

The Standard Model
of Elementary Particle Physics

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This corresponds to roughly the second half of the Standard Model course (MA4447) taught by Stefan Sint in Michælmas term 2011. It picks up where David Whyte's excellent summary (www.maths.tcd.ie/~dawhyte/SM-summary.pdf) left off.

To set the scene, we are investigating gauge fields of $\mathfrak{su}(2)$.

1 Non-Abelian Gauge Fields

$A_\mu(x)$ is, for a fixed μ, x, a Hermitian, traceless, 2×2 matrix. We have a basis $\{T^a,\ a=1,2,3\}$, $T^a = \frac{\tau^a}{2}$ such that

$$(X,Y) = 2\,\mathrm{tr}(XY),$$
$$\left(T^a, T^b\right) = 2\,\mathrm{tr}\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\right) = \delta^{ab},$$
$$X = X^a T^a, \quad X^a = \left(X, T^a\right),$$

for any Hermitian X with $\mathrm{tr}(X) = 0$. Then we can write $A_\mu(x)$ as

$$A_\mu(x) = A_\mu^a(x)\,T^a = A_\mu^a(x)\frac{\tau^a}{2}$$

in SU(2). To determine the field strength, we calculate the commutator $\left[D_\mu, D_\nu\right]$, where $D_\mu = \partial_\mu + igA_\mu$,

$$\begin{aligned}\left[D_\mu, D_\nu\right]\psi(x) &= \left[\partial_\mu + igA_\mu, \partial_\nu + igA_\nu\right]\psi(x)\\ &= \Big\{\left[\partial_\mu, \partial_\nu\right] + ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + igA_\mu\partial_\nu + igA_\nu\partial_\mu\\ &\qquad - igA_\nu\partial_\mu - igA_\mu\partial_\nu - g^2\left[A_\mu, A_\nu\right]\Big\}\psi(x)\\ &= ig\Big\{\partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]\Big\}\psi(x)\\ &= igF_{\mu\nu}(x)\psi(x).\end{aligned}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$

and therefore $F_{\mu\nu}(x)$ is Hermitian and traceless. In SU(2),

$$F_{\mu\nu}(x) = F_{\mu\nu}^a(x)\frac{\tau^a}{2}.$$

To determine $F_{\mu\nu}^a(x)$,

$$\begin{aligned}F_{\mu\nu}^a(x) &= \left(F_{\mu\nu}(x), T^a\right)\\ &= 2\,\mathrm{tr}\left(F_{\mu\nu}(x)\tfrac{\tau^a}{2}\right)\\ &= \partial_\mu\left(A_\nu(x), T^a\right) - \partial_\nu\left(A_\mu(x), T^a\right) + ig\left(\left[A_\mu(x), A_\nu(x)\right], T^a\right)\end{aligned}$$

where we project onto the T^a basis using the linearity of the scalar product.

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + ig\,2\,\mathrm{tr}\left\{\left[A_\mu^b(x)\frac{\tau^b}{2}, A_\nu^c(x)\frac{\tau^c}{2}\right]\frac{\tau^a}{2}\right\}$$

$$= \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + ig2A_\mu^b(x)A_\nu^c(x)\,\mathrm{tr}\left\{\left[\frac{\tau^b}{2},\frac{\tau^c}{2}\right]\frac{\tau^a}{2}\right\}.$$

Noting that

$$\begin{aligned}\mathrm{tr}\left\{\left[\frac{\tau^b}{2},\frac{\tau^c}{2}\right]\frac{\tau^a}{2}\right\} = \mathrm{tr}\left\{i\varepsilon^{bcd}\frac{\tau^d}{2}\frac{\tau^a}{2}\right\} &= i\varepsilon^{bcd}\mathrm{tr}\left(\frac{\tau^d}{2}\frac{\tau^a}{2}\right)\\ = i\varepsilon^{bcd}\frac{1}{2}\delta^{da} = \frac{1}{2}i\varepsilon^{bca} &= \frac{i}{2}\varepsilon^{abc},\end{aligned}$$

for SU(2) we arrive at

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - g\varepsilon^{abc}A_\mu^b(x)A_\nu^c(x), \qquad a = 1,2,3.$$

These are Yang–Mills fields.

We would like to write down a Lagrangian for SU(2) Yang–Mills theory, which must be Lorentz and gauge invariant. We can make a Lorentz invariant $F_{\mu\nu}F^{\mu\nu}$. But how does $F_{\mu\nu}(x)$ transform under local SU(2) (gauge) transformations? We know that

$$A_\mu(x) \to U(x)A_\mu(x)U^{-1}(x) + \frac{1}{ig}U(x)\partial_\mu U^{-1}(x) \equiv A'_\mu(x).$$

From the transformation $A_\mu \to A'_\mu$, we can work out $F_{\mu\nu} \to F'_{\mu\nu}$. However, it is easier to note that

$$F_{\mu\nu}(x) = \frac{1}{ig}\left[D_\mu, D_\nu\right].$$

We know that $\psi(x) \to U(x)\psi(x)$ and $D_\mu\psi(x) \to U(x)D_\mu\psi(x)$ by construction. Therefore

$$\begin{aligned} D_\mu &\to U(x)D_\mu U(x)^{-1},\\ \partial_\mu + igA_\mu(x) &\to U(x)\left(\partial_\mu + igA_\mu(x)\right)U(x)^{-1},\\ D_\mu D_\nu &\to U(x)D_\mu U(x)^{-1}U(x)D_\nu U(x)^{-1} = U(x)D_\mu D_\nu U(x)^{-1},\\ F_{\mu\nu}(x) &\to U(x)F_{\mu\nu}(x)U(x)^{-1} \equiv F'_{\mu\nu}(x),\end{aligned}$$

so unlike the U(1) case, in SU(2) $F_{\mu\nu}$ is not gauge invariant.

To get a Lagrangian density, we need a Lorentz invariant and gauge invariant field. $F_{\mu\nu}(x)F^{\mu\nu}(x)$ is Lorentz invariant but not gauge invariant since under a gauge transformation it becomes $U(x)F_{\mu\nu}(x)F^{\mu\nu}(x)U^{-1}(x)$. Therefore, we will take the trace.

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{tr}\Big[F_{\mu\nu}(x)F^{\mu\nu}(x)\Big],$$

the Yang–Mills Lagrangian density. In components,

$$\begin{aligned}\mathscr{L}_{\mathrm{YM}} &= -\frac{1}{2}\mathrm{tr}\left\{F^a_{\mu\nu}(x)\frac{\tau^a}{2}F^{b\,\mu\nu}(x)\frac{\tau^b}{2}\right\}\\ &= -\frac{1}{2}F^a_{\mu\nu}(x)F^{b\,\mu\nu}(x)\mathrm{tr}\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\right)\\ &= -\frac{1}{4}F^a_{\mu\nu}(x)F^{a\,\mu\nu}(x)\end{aligned}$$

as $\mathrm{tr}\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\right) = \frac{1}{2}\delta^{ab}$.

We can get the SU(2) Yang–Mills field equations by

$$\partial_\mu \frac{\partial \mathscr{L}_{\mathrm{YM}}(x)}{\partial\left(\partial_\mu A^a_\nu(x)\right)} = \frac{\partial \mathscr{L}_{\mathrm{YM}}(x)}{\partial A^a_\nu(x)} \qquad (a = 1,2,3,\ \nu = 0,1,2,3).$$

These generalise the Maxwell equations. We can now couple a fermion or boson to the Yang–Mills field.

The Lagrangian for a fermion and a boson coupled to a SU(2) YM field is

$$\begin{aligned}\mathscr{L}(x) = &-\frac{1}{4}F^a_{\mu\nu}(x)F^{a\,\mu\nu}(x) + \bar{\psi}(x)\,(i\not{D} + m)\,\psi(x)\\ &+ \left(D_\mu\phi(x)\right)^\dagger D^\mu\phi(x) + m^2\phi^\dagger(x)\phi(x)\end{aligned}$$

where $\psi = \binom{\psi_1}{\psi_2}$, $\phi = \binom{\phi_1}{\phi_2}$ and 1, 2 are flavour indices. These transform as

$$\phi(x) \to U(x)\phi(x), \qquad \psi(x) \to U(x)\psi(x).$$

Most of this is not specific to SU(2). Only small changes are needed to describe arbitrary SU(N):

$$\left[T^a, T^b\right] = if^{abc}T^c.$$

What is the dimension of the space of $N \times N$ Hermitian matrices with trace 0?

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & & \\ b^*_{12} & b_{22} & & & \\ \vdots & & \ddots & & \vdots \\ & & & b_{N-1,N-1} & b_{N-1,N} \\ & & \cdots & b^*_{N-1,N} & b_{NN}\end{pmatrix}.$$

There are $N-1$ real numbers on the diagonal and N^2 elements. As $B = B^\dagger$, $\mathrm{tr}(B) = 0$ and $N^2 - 1$ real numbers can be picked freely. For example,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b^*_{12} & b_{22} & b_{23} \\ b^*_{13} & b^*_{23} & b_{33}\end{pmatrix}$$

gives us 8 real numbers. Our basis is $\left\{T^a\right\}_{a=1,\dots,N^2-1}$.

2 The QCD Lagrangian Density

For a quark u,

$$\psi_u(x) = \begin{pmatrix} \psi_u(x)_{\text{blue}} \\ \psi_u(x)_{\text{green}} \\ \psi_u(x)_{\text{red}} \end{pmatrix}.$$

We will use $i = 1, 2, 3$ instead of blue, green, red.

For the ψ_u above, we have a local symmetry

$$\psi_u(x) \to U(x)\psi_u(x) = \psi'_u(x).$$

In index notation,

$$\psi'_u(x)_i = \sum_{j=1}^{3} U_{ij}(x)\psi_u(x)_j, \qquad U(x) \in \text{SU}(3)_{\text{colour}}.$$

Our Lagrangian is now

$$\mathscr{L}_{\text{QCD}}(x) = -\frac{1}{2}\text{tr}\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right) + \sum_f \bar{\psi}_f(x)\left(i\not{D} - m_f\right)\psi_f(x)$$

for $f = u, d, s, c, b, t$. In component form,

$$\mathscr{L}_{\text{quark}}(x) = \sum_f \sum_{\alpha,\beta=1}^{4} \sum_{i,j=1}^{3} \bar{\psi}_f(x)_{\alpha i}\left(i\gamma^\mu_{\alpha\beta}\left[\partial_\mu\delta_{ij} + ig\sum_{a=1}^{8} A^a_\mu(x)T^a_{ij}\right] - m_f\delta_{\alpha\beta}\delta_{ij}\right)\psi_f(x)_{\beta j}.$$

We are summing over flavour, spinor (Dirac) and colour (SU(3)) indices.

2.1 Symmetries of the QCD Lagrangian

- We have an exact local $\text{SU}(3)_{\text{colour}}$ invariance (by construction).
- Gluons are "flavour blind", i.e. gluons couple to all flavours equally.

We will introduce more compact notation:

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \vdots \\ \psi_t \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{\psi}_u, \bar{\psi}_d, \ldots, \bar{\psi}_t\right),$$

$$M = \text{diag}(m_u, m_d, \ldots, m_t),$$

$$\mathscr{L}_{\text{quark}} = \bar{\psi}(i\not{D} - M)\psi.$$

If all masses are equal, $m_u = m_d = \cdots = m_t = m$,

$$\bar{\psi}\left(i\not{D} - M\right)\psi = \bar{\psi}\left(i\not{D} - m\right) \otimes \mathbb{1}_6 \psi.$$

This implies a global invariance under U(6) transformations (and hence SU(6)).

$$\psi_f \to \sum_g U_{fg}\psi_g; \qquad \bar{\psi}_f \to \sum_g \bar{\psi}_g U^{-1}_{gf}$$

or

$$\psi \to U\psi, \qquad \bar{\psi} \to \bar{\psi}U^{-1}, \qquad U \in \mathrm{U}(6) = \mathrm{U}(1) \times \mathrm{SU}(6).$$

Experimentally, $m_u \approx m_d = \mathcal{O}(1 \text{ MeV})$,

$$\frac{m_s}{\frac{1}{2}\left(m_u + m_d\right)} \approx 24,$$
$$m_c \approx 1.3 \text{ GeV},$$
$$m_b \approx 45 \text{ GeV},$$
$$m_t \approx 176 \text{ GeV}.$$

How do you weigh a quark? These values are conventional and in some way depend on the renormalisation scheme. Hadronisation is inherently non–perturbative.

If we take only two flavours and set the masses equal, then

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}, \qquad M = \mathrm{diag}(m_u, m_d) = \mathrm{diag}(m, m)$$

and we have a global invariance under SU(2) transformation. This is $\mathrm{SU}(2)_{\mathrm{isospin}}$.

Historically, $\mathrm{SU}(2)_{\mathrm{isospin}}$ is an approximate symmetry between the proton and neutron, where the proton is the bound state of uud and the neutron is the bound state of udd.

If we include the strange quark, $m_u = m_d = m_s$, there is an exact global $\mathrm{SU}(3)_{\mathrm{flavour}}$ symmetry (introduced by Gellmann). This leads to the classification of the hadron spectrum as irreducible representations of this group.

2.2 Global Chiral Symmetry

Consider projectors $P_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right)$. The projector properties are

$$P_\pm^\dagger = \frac{1}{2}\left(1 \pm \gamma_5^\dagger\right) = \frac{1}{2}\left(1 \pm \gamma_5\right) = P_\pm$$
$$P_\pm^2 = \frac{1}{4}\left(1 \pm \gamma_5\right)\left(1 \pm \gamma_5\right) = \frac{1}{4}\left(1 \pm 2\gamma_5 + \gamma_5^2\right) = \frac{1}{4}\left(2 \pm 2\gamma_5\right) = \frac{1}{2}\left(1 \pm \gamma_5\right) = P_\pm$$
$$P_+ + P_- = \frac{1}{2}\left(1 + \gamma_5\right) + \frac{1}{2}\left(1 - \gamma_5\right) = 1$$

$$P_+P_- = P_+(1-P_+) = P_+ - P_+^2 = P_+ - P_+ = 0$$
$$P_-P_+ = 0.$$

Consider N_f massless fermions,

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_f} \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{\psi}_1, \dots, \bar{\psi}_{N_f}\right)$$

Then

$$\begin{aligned} \mathscr{L}_{\text{quarks}} &= \bar{\psi} \not{D} \psi \\ &= \bar{\psi} \underbrace{(P_+ + P_-)}_{\mathbb{1}} \not{D} \underbrace{(P_+ + P_-)}_{\mathbb{1}} \psi \\ &= \bar{\psi} P_+ \not{D} P_- \psi + \bar{\psi} P_- \not{D} P_+ \psi \end{aligned}$$

as P_+P_- terms disappear. Here we used $\gamma^\mu P_\pm = P_\mp \gamma^\mu$ since $\gamma^\mu\gamma^5 = \gamma^5\gamma^\mu$, which implies $P_+\gamma^\mu P_+ = P_+P_-\gamma^\mu = 0$.

Recall the QCD Lagrangian

$$\mathscr{L}_{\text{QCD}}(x) = -\frac{1}{2}\text{tr}\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right) + \sum_f \bar{\psi}_f(x)\left(i\not{D} - m_f\right)\psi_f(x)$$

where $\not{D} = \gamma^\mu\left(\partial_\mu + igA_\mu(x)\right)$. We have exact SU(3) colour symmetry ($U(x) \in$ SU(3)),

$$\psi_f(x) \to U(x)\psi_f(x), \qquad \bar{\psi}_f(x) \to \bar{\psi}_f(x)U(x)^{-1}, \qquad U(x) \in \text{SU}(3).$$

Interaction between gluons and quarks is the same for all flavours ("flavour blindness"). There is an approximate global $\text{SU}(N_f)_{\text{flavour}}$ symmetry,

$$\begin{aligned} &N_f = 2, \qquad m_u = m_d = m, && \text{(isospin symmetry)} \\ &N_f = 3, \qquad m_u = m_d = m_s = m, && \text{("eightfold way" $\sim$ classification of hadrons).} \end{aligned}$$

We also have parity symmetry.

$$\begin{gathered} P : x^\mu \to \tilde{x}^\mu = \left(x^0, -\mathbf{x}\right) = (-1)^{1+\delta_{\mu 0}} x^\mu, \\ \psi(x) \to \gamma^0\psi(\tilde{x}), \qquad \bar{\psi}(x) \to \bar{\psi}(\tilde{x})\gamma^0, \\ A_\mu(x) \to (-1)^{1+\delta_{\mu 0}} A_\mu(\tilde{x}) \end{gathered}$$

(vector fields transform like x^μ),

$$\bar{\psi}(x)\gamma^\mu\left(\partial_\mu + igA_\mu(x)\right)\psi(x) \xrightarrow{P} \bar{\psi}(\tilde{x})\gamma^0\gamma^\mu\left(\partial_\mu + ig(-1)^{1+\delta_{\mu 0}}A_\mu(\tilde{x})\right)\gamma^0\psi(\tilde{x}).$$

The partial derivatives transform too:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\nu}{\partial x^\mu} = \sum_\nu \delta_{\nu\mu}(-1)^{1+\delta_{\nu 0}}\frac{\partial}{\partial \tilde{x}^\nu} = (-1)^{1+\delta_{\nu 0}}\frac{\partial}{\partial \tilde{x}^\nu}.$$

As

$$\gamma^0\gamma^\mu = \left\{ \begin{array}{cl} \gamma^\mu\gamma^0 & \text{if } \mu = 0 \\ -\gamma^\mu\gamma^0 & \text{if } \mu = k = 1,2,3 \end{array} \right\} = -\gamma^\mu\gamma^0(-1)^{1+\delta_{\mu 0}},$$

so,

$$\begin{aligned} &\bar{\psi}(x)\gamma^\mu\left(\partial_\mu + igA_\mu(x)\right)\psi(x) \\ &\xrightarrow{P} \bar{\psi}(\tilde{x})\underbrace{(-1)^{2(1+\delta_{\mu 0})}}_{\mathbb{1}}\gamma^\mu\left((-1)^{1+\delta_{\mu 0}}\frac{\partial}{\partial \tilde{x}^\mu} + ig(-1)^{1+\delta_{\mu 0}}A_\mu(\tilde{x})\right)\psi(\tilde{x}) \\ &= \bar{\psi}(\tilde{x})\gamma^\mu\left(\frac{\partial}{\partial \tilde{x}^\mu} + igA_\mu(\tilde{x})\right)\psi(\tilde{x}). \end{aligned}$$

Therefore $\mathscr{L}_{\text{QCD}}(x) \xrightarrow{P} \mathscr{L}_{\text{QCD}}(\tilde{x})$. Note that $\tilde{x}^1 = -x^1$, $\mathrm{d}\tilde{x}^1 = -\mathrm{d}x^1$, so

$$\int_{-R}^{R}\mathrm{d}x^1 = -\int_{R}^{-R}\mathrm{d}\tilde{x}^1 = \int_{-R}^{R}\mathrm{d}\tilde{x}^1.$$

Thus, the action

$$\int \mathrm{d}^4x\, \mathscr{L}_{\text{QCD}}(x) \xrightarrow{P} \int \mathrm{d}^4\tilde{x}\, \mathscr{L}_{\text{QCD}}(\tilde{x})$$

is parity invariant (in contrast to weak interactions).

2.3 Chiral Symmetry of Massless Fermions

We have $P_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right)$, the chiral projectors. Recall the projector properties: $P_\pm^\dagger = P_\pm$ (Hermitian); $P_+ + P_- = \mathbb{1}$; $P_\pm^2 = P_\pm$; $P_+P_- = 0$. For massless fermions,

$$\begin{aligned} \bar{\psi}i\partial\!\!\!/\psi &= \bar{\psi}i(P_+ + P_-)\partial\!\!\!/(P_+ + P_-)\psi \\ &= \bar{\psi}i\left(P_+\partial\!\!\!/P_+ + P_+\partial\!\!\!/P_- + P_-\partial\!\!\!/P_+ + P_-\partial\!\!\!/P_-\right)\psi. \end{aligned}$$

Note that $\left\{\gamma_5, \gamma^\mu\right\} = 0$, which implies that

$$P_\pm\gamma^\mu = \gamma^\mu P_\mp, \qquad P_\pm\gamma^\mu P_\pm = \gamma^\mu P_\mp P_\pm = 0.$$

Looking at the Dirac term,

$$\begin{aligned} \bar{\psi}i\partial\!\!\!/\psi &= \bar{\psi}P_+i\partial\!\!\!/P_-\psi + \bar{\psi}P_-i\partial\!\!\!/P_+\psi \\ &= \bar{\psi}_Li\partial\!\!\!/\psi_L + \bar{\psi}_Ri\partial\!\!\!/\psi_R. \end{aligned}$$

This is like we have two independent fields which are not coupled. A mass term couples left and right fields:

$$\begin{aligned} m\bar{\psi}\psi &= m\bar{\psi}(P_+ + P_-)\psi = m\left(\bar{\psi}P_+\psi + \bar{\psi}P_-\psi\right) \\ &= m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right). \end{aligned}$$

As an aside:

$$\bar{\psi}_L = \psi_L^\dagger\gamma^0 = \left(P_+\psi\right)^\dagger\gamma_0 = \psi^\dagger P_-\gamma_0 = \psi^\dagger\gamma_0 P_+ = \bar{\psi}P_+,$$

but $\psi_L = P_-\psi$.

If $m = 0$, left and right handed fields can be transformed independently. If $\psi = \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_N\end{pmatrix}$, then

$$\begin{aligned} &\psi_L \to U_L\psi_L; \qquad \bar{\psi}_L \to \bar{\psi}_L U_L^{-1}, \qquad U_L \in \mathrm{SU}(N)_L, \\ &\psi_R \to U_R\psi_R; \qquad \bar{\psi}_R \to \bar{\psi}_R U_R^{-1}, \qquad U_R \in \mathrm{SU}(N)_R. \end{aligned}$$

We have a global chiral symmetry $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R \ni (U_L, U_R)$.

Weak interactions are based on a local $\mathrm{SU}(2)_L$ symmetry. This means the fermions are grouped in left-handed doublets.

$$\begin{matrix} \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}_L & \begin{pmatrix}\psi_s \\ \psi_c\end{pmatrix}_L & \begin{pmatrix}\psi_t \\ \psi_b\end{pmatrix}_L \\ \begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L & \begin{pmatrix}\psi_{\nu_\mu} \\ \psi_\mu\end{pmatrix}_L & \begin{pmatrix}\psi_{\nu_\tau} \\ \psi_\tau\end{pmatrix}_L \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} \end{matrix}$$

3 Weak Interactions, $\mathbf{SU(2)}_L \times \mathbf{U(1)}_Y$ Gauge Symmetry

Consider the first family, only leptons. The free Lagrangian density is

$$\mathscr{L}_0(x) = \left(\bar{\psi}_{\nu_e}(x), \bar{\psi}_e(x)\right)_L i\partial\!\!\!/ \begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L + \bar{\psi}_{e,R}(x)\, i\partial\!\!\!/ \psi_{e,R}(x).$$

Here we made the assumption that ν_e is left-handed and hence massless. Recent experiments indicate that this is not true. Under $\mathrm{SU}(2)_L$ symmetry,

$$\begin{aligned} &\begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L \to U_L\begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L, \qquad \left(\bar{\psi}_{\nu_e}, \bar{\psi}_e\right)_L \to \left(\bar{\psi}_{\nu_e}, \bar{\psi}_e\right)_L U_L^{-1}, \\ &\psi_{e,R} \to \psi_{e,R}, \qquad \bar{\psi}_{e,R} \to \bar{\psi}_{e,R}. \end{aligned}$$

Right-handed fields transform trivially: they are singlets under $\mathrm{SU}(2)_L$.

We now promote $\mathrm{SU}(2)_L$ to a local symmetry (gauge it), $U_L \to U_L(x)$, and introduce the $\mathrm{SU}(2)_L$ gauge field

$$W_\mu(x) = W_\mu^a(x)\frac{\tau^a}{2}$$

with field tensor

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig\left[W_\mu, W_\nu\right].$$

Our interaction Lagrangian becomes

$$\begin{aligned}\mathscr{L}(x) = \left(\bar\psi_{\nu_e}(x), \bar\psi_e(x)\right)_L i\gamma^\mu \left(\partial_\mu + igW_\mu^a(x)\frac{\tau^a}{2}\right)\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L \\ + \bar\psi_{e,R}(x) i\partial\!\!\!/ \psi_{e,R}(x) - \frac{1}{2}\mathrm{tr}\left(W_{\mu\nu}(x)W^{\mu\nu}(x)\right).\end{aligned}$$

Defining $W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right)$, the $\nu - e - W$ interaction is

$$\begin{aligned}\mathscr{L}'(x) &= -g\left(\bar\psi_{\nu_e}, \bar\psi_e\right)_L \gamma^\mu W_\mu^a \frac{\tau^a}{2}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L \\ &= -g\left(\bar\psi_{\nu_e}, \bar\psi_e\right)_L \gamma^\mu \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3\end{pmatrix}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L \\ &= -\frac{g}{2}\Big\{W_\mu^3\left[\bar\psi_{\nu_e,L}\gamma^\mu\psi_{\nu_e,L} - \bar\psi_{e,L}\gamma^\mu\psi_{e,L}\right] \\ &\qquad + \sqrt{2}W_\mu^+\bar\psi_{\nu_e,L}\gamma^\mu\psi_{e,L} + \sqrt{2}W_\mu^-\bar\psi_{e,L}\gamma^\mu\psi_{\nu_e,L}\Big\}.\end{aligned}$$

Notice that

$$\begin{aligned}\bar\psi_{e,L}\gamma^\mu\psi_{\nu_e,L} &= \bar\psi_e P_+ \gamma^\mu P_- \psi_{\nu_e} \\ &= \bar\psi_e \gamma^\mu P_- \psi_{\nu_e} \\ &= \frac{1}{2}\bar\psi_e\gamma^\mu\left(1-\gamma_5\right)\psi_{\nu_e} \\ &= \frac{1}{2}\Big[\underbrace{\bar\psi_e\gamma^\mu\psi_{\nu_e}}_{\text{vector current}} - \underbrace{\bar\psi_e\gamma^\mu\gamma_5\psi_{\nu_e}}_{\text{axial vector current}}\Big]. \\ &\qquad\qquad V \qquad - \qquad A\end{aligned}$$

This has decay $\mu^- \to e^- + \bar\nu_e + \nu_\mu$.

We are still missing some ingredients, i.e. mass. Massless $\Rightarrow$ Coulomb $\Rightarrow$ long range, but the weak interaction is short range. For now, we will ignore this problem.

How do we describe electromagnetism? W_μ^3 cannot describe a photon since it couples to $\nu - \bar{\nu}$ and does not couple to $\psi_{e,R}$.

Reconsider $\mathscr{L}_0(x)$. There are two more global U(1) symmetries:

$$\begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L \to e^{i\varphi} \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L, \qquad \psi_{e,R} \to e^{i\varphi'} \psi_{e,R}.$$

φ and φ' are two independent parameters. To incorporate the photon, we need to gauge only one combination. We choose

$$\begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L \to e^{iy_L\chi} \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L, \qquad \psi_{e,R} \to e^{iy_R\chi} \psi_{e,R}.$$

y_L, y_R are fixed numbers to be determined later. Define

$$\psi = \begin{pmatrix} \psi_{\nu_e,L} \\ \psi_{e,L} \\ \psi_{e,R} \end{pmatrix} \to e^{i\chi Y} \begin{pmatrix} \psi_{\nu_e,L} \\ \psi_{e,L} \\ \psi_{e,R} \end{pmatrix} = e^{i\chi Y}\psi, \qquad Y = \begin{pmatrix} y_L & 0 & 0 \\ 0 & y_L & 0 \\ 0 & 0 & y_R \end{pmatrix}.$$

Y is a generator of the $\mathrm{U}(1)_Y$ group, the weak hypercharge. Promote $\mathrm{U}(1)_Y$ to a local symmetry, i.e. $\chi = \chi(x)$. In order to maintain invariance, we introduce the Abelian gauge field B_μ with field tensor

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

Now, including the W field as well,

$$\mathscr{L}(x) = \frac{1}{2}\mathrm{tr}\left(W_{\mu\nu}(x) W^{\mu\nu}(x)\right) - \frac{1}{4} B_{\mu\nu}B^{\mu\nu} + \bar{\psi}(x) i\gamma^\mu D_\mu \psi(x),$$
$$D_\mu\psi(x) = \left(\partial_\mu + igW_\mu^a(x)T^a + ig'B_\mu(x)Y\right)\psi(x)$$

where

$$T^a = \begin{pmatrix} \frac{\tau^a}{2} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\frac{\tau^a}{2}$ is a 2×2 matrix. T^a is the fundamental representation for the doublet and trivial for the singlet.

$$\left[T^a, T^b\right] = i\varepsilon^{abc}T^c, \qquad \left[T^a, Y\right] = 0.$$

The interaction part of the Lagrangian is

$$\mathscr{L}' = -\bar{\psi}\gamma^\mu \left(gW_\mu^a T^a + ig'B_\mu Y\right)\psi$$

$$
\begin{aligned}
&= -\frac{g}{\sqrt{2}}\left[W_\mu^+\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{e,L} + W_\mu^-\bar{\psi}_{e,L}\gamma^\mu\psi_{\nu_e,L}\right] \\
&\quad -\frac{1}{2}\left[gW_\mu^3 + 2y_L g' B_\mu\right]\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{\nu_e,L} \\
&\quad +\frac{1}{2}\left[gW_\mu^3 - 2y_L g' B_\mu\right]\bar{\psi}_{e,L}\gamma^\mu\psi_{e,L} \\
&\quad - y_R g' B_\mu \bar{\psi}_{e,R}\gamma^\mu\psi_{e,R}.
\end{aligned}
$$

Either y_L or y_R can be fixed arbitrarily, since only $g' y_L$ and $g' y_R$ appear. Set $y_L = \frac{1}{2}$ and define the Z-boson and photon field as

$$
Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gW_\mu^3 - g'B_\mu\right), \qquad A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'W_\mu^3 + gB_\mu\right).
$$

Define

$$
\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \qquad \cos\theta_W = \frac{g}{\sqrt{g'^2+g^2}}.
$$

Then

$$
\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}.
$$

This matrix is easily invertible. θ_W is the weak mixing angle.

How is $\mathrm{U}(1)_{\mathrm{EM}}$ embedded? Putting Z_μ, A_μ into the Lagrangian,

$$
\begin{aligned}
\mathscr{L} = -\sqrt{g^2+g'^2}\,Z_\mu\Big\{&\frac{1}{2}\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{\nu_e,L} - \frac{1}{2}\bar{\psi}_{e,L}\gamma^\mu\psi_{e,L} \\
&- \sin^2\theta_W\left(-\bar{\psi}_{e,L}\gamma^\mu\psi_{e,L} + y_R\bar{\psi}_{e,R}\gamma^\mu\psi_{e,R}\right)\Big\} \\
-\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu\Big\{&-\bar{\psi}_{e,L}\gamma^\mu\psi_{e,L} + y_R\bar{\psi}_{e,R}\gamma^\mu\psi_{e,R}\Big\}.
\end{aligned}
$$

Notice that $\bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R = \bar{\psi}\gamma^\mu\psi$ implies $y_R = -1$, $\frac{gg'}{\sqrt{g^2+g'^2}} = e$.

$$
\mathscr{L} = \cdots + eA_\mu j^\mu_{\mathrm{EM}}, \qquad j^\mu_{\mathrm{EM}} = \bar{\psi}_e\gamma^\mu\psi_e
$$

which is the usual term we would expect. j^μ_{EM} is the conserved Noether current of $\mathrm{U}(1)$ symmetry. It follows that

$$
\begin{aligned}
e = g\sin\theta_W = g'\cos\theta_W &= \sqrt{g^2+g'^2}\sin\theta_W\cos\theta_W, \\
\psi_{\substack{L\\R}} = P_\mp\psi &= \frac{1}{2}\left(1\mp\gamma_5\right)\psi,
\end{aligned}
$$

$$\begin{aligned}\bar{\psi}\gamma^\mu\psi &= \bar{\psi}\,(P_+ + P_-)\,\gamma^\mu\,(P_+ + P_-)\,\psi \\ &= \bar{\psi}P_+\gamma^\mu P_-\psi + \bar{\psi}P_-\gamma^\mu P_+\psi \\ &= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R.\end{aligned}$$

Tidying up the Lagrangian,

$$\begin{aligned}\mathscr{L} = \cdots + eA_\mu j^\mu_{\mathrm{EM}} &- \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu j^\mu_{\mathrm{NC}} \\ &- \frac{e}{\sqrt{2}\sin\theta_W}\left(W^+_\mu\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{e,L} + W^-_\mu\bar{\psi}_{e,L}\gamma^\mu\psi_{\nu_e,L}\right),\end{aligned}$$

where

$$j^\mu_{\mathrm{NC}} = \frac{1}{2}\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{\nu_e,L} - \frac{1}{2}\bar{\psi}_{e,L}\gamma^\mu\psi_{e,L} + \sin^2\theta_W j^\mu_{\mathrm{EM}}.$$

j^μ_{NC} is the neutral current density. A_μ, Z_μ are neutral particles, $W^\pm_\mu$ carry an electric charge.

Consider $\mathscr{L}(\phi, \partial_\mu\phi)$, $\phi = \begin{pmatrix}\phi_1 \\ \vdots \\ \phi_N\end{pmatrix}$, $\phi = \phi' = R\phi$, $R \in \mathrm{O}(N) \simeq (\mathbb{1} + i\alpha^a T^a)\,\phi$.

$$\begin{aligned}\delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi^a(x)}\delta\phi + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^a(x)\right)}\delta\partial_\mu\phi, \qquad \phi + \delta\phi = i\alpha^a T^a\phi, \\ &= \frac{\partial\mathscr{L}}{\partial\phi^a(x)}i\alpha^a T^a\phi + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^a(x)\right)}\partial_\mu i\alpha^a T^a\phi\end{aligned}$$

etc., and

$$0 = \int \mathrm{d}^3\mathbf{x}\,\partial^\mu j^a_\mu(x) = \int \mathrm{d}^3\mathbf{x}\,\partial^0 j^a_0(x) = \partial^0\int \mathrm{d}^3\mathbf{x}\, j^a_0(x) = \partial^0 Q^a$$

as we loose terms at infinity, so we forget about the $\partial^i j_i$ term since boundary terms vanish.

Bosons couple to the Noether current of the symmetry. Take muon decay, $\bar{\mu} \to \bar{e} + \bar{\nu}_e + \nu_\mu$, $L_\mu = 1$ and lepton number is a charge conserved in each family individually.

In U(2) symmetry, $\mathrm{U}(2) = \underbrace{\mathrm{SU}(2)}_{\text{local}} \times \underbrace{\mathrm{U}(1)}_{\text{global}}$. We have global symmetry for these weak doublets,

$$\begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L \to e^{i\alpha}\begin{pmatrix}\psi_{\nu_e} \\ \psi_e\end{pmatrix}_L, \qquad \left(\bar{\psi}_{\nu_e}, \bar{\psi}_e\right)_L \to \left(\bar{\psi}_{\nu_e}, \bar{\psi}_e\right)_L e^{-i\alpha}.$$

The separately conserved currents for global U(1) symmetry provide a lepton number for each family. However, neutrino oscillations mean this is not an exact symmetry—neutrinos have mass! For independent symmetries choose a different α for each family (3 symmetries). This gives a conserved charge—the lepton number. We recover exact symmetries in the limit $m_\nu \to 0$. There is indirect evidence of symmetry breaking in neutrino mixing.

4 The Higgs Sector of the Standard Model

Since the photon is massless, the Coulomb law is valid (and vice versa). Our goal is to introduce mass terms for $W_\mu^\pm$ and Z_μ (but not the photon) and mass terms for fermions. Both break $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ gauge invariance. For example, the mass term for Z_μ: $\sim \frac{1}{2}m_Z^2 Z_\mu Z^\mu$; and for the electron: $\sim \bar{\psi}_e\psi_e = \bar{\psi}_{e,L}\psi_{e,R}+\bar{\psi}_{e,R}\psi_{e,L}$. These are not invariant under the $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ transformation.

Spontaneous symmetry breaking comes to the rescue! Symmetries of the Lagrangian do not equal symmetries of the vacuum state.

4.1 Higgs Sector

Introduce a complex scalar doublet

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad \mathscr{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi), \qquad V(\phi) = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right).$$

For $\mu^2 < 0$ the minimum of the potential is at $|\phi| = \frac{v}{\sqrt{2}}$, implying SSB.

What are the symmetries of $\mathscr{L}$?

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\varphi_1 + i\varphi_3\\ \varphi_2 + i\varphi_4\end{pmatrix}$$

and rearranging as an O(4) vector,

$$\vec{\varphi} = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} \quad\Rightarrow\quad \mathscr{L} = \frac{1}{2}\partial_\mu\vec{\varphi}^\dagger\partial^\mu\vec{\varphi} - \tilde{V}(\vec{\varphi}),$$

$$\tilde{V}(\vec{\varphi}) = \frac{1}{2}\mu^2|\vec{\varphi}|^2 + \frac{\lambda}{4}|\vec{\varphi}|^4, \qquad |\vec{\varphi}|^2 = \varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2.$$

We have a global symmetry,

$$\vec{\varphi} \to R\vec{\varphi}, \qquad R \in \mathrm{O}(4),$$
$$\vec{\varphi}^T\vec{\varphi} \to \vec{\varphi}^T R^T R\vec{\varphi} = \vec{\varphi}^T\vec{\varphi}$$

as $R^T R = \mathbb{1}$ for $R \in \mathrm{O}(4)$. If one component takes a non-zero vacuum expectation value, we have a subgroup of the symmetry.

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & & & \\ 0 & & \bar{R} & \\ 0 & & & \end{pmatrix}\begin{pmatrix}v\\0\\0\\0\end{pmatrix} = \begin{pmatrix}v\\0\\0\\0\end{pmatrix}, \qquad \bar{R} \in \mathrm{O}(3)$$

where $\vec{\varphi}^T \vec{\varphi} = v$ is the ground state for the SSB potential. Thus, under spontaneous symmetry breaking, O(4)(symmetries of $\mathscr{L}$)→O(3)(symmetries of the ground state). As

$$\mathrm{O}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2),$$

there is a formalism in terms of SU(2) × SU(2) symmetries. Define a 2 × 2 matrix field

$$\frac{1}{\sqrt{2}}\Sigma = \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}, \qquad \begin{aligned} \mathrm{tr}\left(\Sigma^\dagger \Sigma\right) &= 2\,\mathrm{tr}\left[\begin{pmatrix} \phi_2 & \phi_1 \\ -\phi_1^* & \phi_2^* \end{pmatrix}\begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}\right] \\ &= 2\,\mathrm{tr}\begin{pmatrix} \phi_2^*\phi_2 + \phi_1^*\phi_1 & 0 \\ 0 & \phi_1^*\phi_1 + \phi_2^*\phi_2 \end{pmatrix} \\ &= 4\left(\phi_1^*\phi_1 + \phi_2^*\phi_2\right) = 4\phi^\dagger\phi. \end{aligned}$$

The Lagrangian for the Higgs sector is now

$$\mathscr{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) - \frac{\mu^2}{4}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left(\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\right)^2$$

and we have the global symmetry

$$\Sigma \to U_L \Sigma U_R^\dagger, \qquad \Sigma \to U_R \Sigma^\dagger U_L^\dagger, \qquad (U_L, U_R) \in \mathrm{SU}(2) \times \mathrm{SU}(2),$$
$$\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) \to \mathrm{tr}\left(U_R \Sigma^\dagger U_L^\dagger U_L \Sigma U_R\right) = \mathrm{tr}\left(U_R^\dagger U_R \Sigma^\dagger \Sigma\right) = \mathrm{tr}\left(\Sigma^\dagger\Sigma\right).$$

The ground state is

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\Bigg|_{\text{vacuum}} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix},$$

so

$$\begin{aligned} \Sigma_0 &= v\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Sigma_0 &\to U_L \Sigma_0 U_R^\dagger \\ &= v U_L U_R^\dagger \\ &\neq \Sigma_0 \end{aligned}$$

unless $U_L = U_R$. Σ_0 is invariant under the diagonal SU(2) subgroup $\mathrm{SU}(2)_{L+R}$, obtained for $U_L = U_R$. Under SSB,

$$\underbrace{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}_{\simeq \mathrm{O}(4)} \to \underbrace{\mathrm{SU}(2)_{L+R}}_{\simeq \mathrm{O}(3)}.$$

$SU(2)_{L+R}$ is known as "custodial SU(2) symmetry". If we embed local $SU(2)_L \times U(1)_Y$ symmetry,

$$\begin{aligned}\partial_\mu \phi \to D_\mu \phi &= \left(\partial_\mu + igW_\mu + \tfrac{1}{2} ig' B_\mu\right)\phi \\ &\to D_\mu \Sigma = \partial_\mu \Sigma + igW_\mu \Sigma - ig'\Sigma B_\mu \tfrac{\tau^3}{2}\end{aligned}$$

(taking $y_H = \frac{1}{2}$).

Assume that $V(\vec{\varphi})$ is invariant under

$$\vec{\varphi} \to R\vec{\varphi} \simeq \left(\mathbb{1} + i\alpha_a T^a\right)\vec{\varphi}$$

where R is an element of the symmetry group G and T^a are the generators of G, $a = 1, \ldots, n_G$. Invariance implies $V(R\vec{\varphi}) = V(\vec{\varphi})$. Assume that $V(\vec{\varphi})$ has a set of degenerate minima. Pick one spontaneously: $\vec{\varphi} = \vec{v}$.

Consider fluctuations around the minimum $\vec{v}$,

$$\left.\frac{\partial V}{\partial \varphi_i}\right|_{\vec{\varphi}=\vec{v}} = 0, \qquad \left.\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right|_{\vec{\varphi}=\vec{v}} = M^2_{ij}.$$

Eigenvalues of M^2_{ij} are the squared masses of the physical particle excitations around the vacuum.

Assume that the vacuum $\vec{v}$ (the minimum of $V(\vec{\varphi})$) is invariant under the subgroup $H \subset G$ with generators T^b, $b = 1, \ldots, n_H$ ($n_H < n_G$).

$$\left(\mathbb{1} + \alpha_b T^b\right)\vec{v} = \vec{v} \quad \Rightarrow \quad T^b \vec{v} = 0, \qquad b = 1, \ldots, n_H.$$

Invariance of $V(\vec{\varphi})$ under G implies

$$\begin{aligned}0 &= V(R\vec{\varphi}) - V(\vec{\varphi}) \\ &= V\left(\vec{\varphi} + \alpha_a T^a \vec{\varphi}\right) - V(\vec{\varphi}) \\ &= V(\vec{\varphi}) + \frac{\partial V}{\partial \varphi_i} i\alpha_a T^a_{ij}\varphi_j - V(\vec{\varphi}).\end{aligned}$$

Differentiating with respect to φ_k and evaluating at $\vec{\varphi} = \vec{v}$,

$$\begin{aligned}0 &= \left[\frac{\partial V}{\partial \varphi_i} i\alpha_a T^a_{ij}\delta_{ik} + \frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i} i\alpha_a T^a_{ij}\varphi_j\right]_{\vec{\varphi}=\vec{v}} \\ &= \left.\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}\right|_{\vec{\varphi}=\vec{v}} i\alpha_a T^a_{ij} v_j \\ &= M^2_{ij} i\alpha_a T^a_{ij} v_j,\end{aligned}$$

therefore

$$M^2_{ki}\left(T^a\vec{v}\right)_i = 0, \qquad a = 1,\dots,n_G. \tag{$\star$}$$

For the subgroup $H \subset G$, $T^b\vec{v} = 0$ and $(\star)$ holds trivially.

The remaining generators correspond to

$$a \in \{n_H + 1,\dots,n_G\}.$$

$T^a\vec{v}$ is an eigenvector of M^2 with eigenvalue 0. Therefore, we have $n_G - n_H$ massless particles: Goldstone bosons!

Returning to the Higgs sector, we have a doublet

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_3 + i\varphi_1\\ \varphi_4 + i\varphi_2\end{pmatrix} \to \vec{\varphi} = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix}$$

and

$$\mathscr{L} = \frac{1}{2}\partial_\mu\vec{\varphi}^T\partial^\mu\vec{\varphi} - V(\vec{\varphi}), \qquad V(\vec{\varphi}) = \frac{1}{2}\mu^2|\vec{\varphi}|^2 + \frac{\lambda}{4}|\vec{\varphi}|^4,$$

with

$$\begin{aligned}\frac{\partial V}{\partial\varphi_i} &= \frac{\partial}{\partial\varphi_i}\left\{\frac{1}{2}\mu^2\sum_k\varphi_k^2 + \frac{\lambda}{4}\left(\sum_k\varphi_k^2\right)^2\right\}\\ &= \mu^2\varphi_i + \lambda\left(\sum_k\varphi_k^2\right)\varphi_i.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial\varphi_i}V(\vec{\varphi})\bigg|_{\vec{\varphi}=\vec{v}} = 0 &= \mu^2 v_i + \lambda\vec{v}^2 v_i\\ &= v_i\left(\mu^2 + \lambda\vec{v}^2\right)\\ &\iff \quad v_i = 0 \quad \text{or} \quad \vec{v}^2 = -\frac{\mu^2}{\lambda}.\end{aligned}$$

$\lambda > 0$ is required for stability, so $\vec{v}^2 = -\frac{\mu^2}{\lambda}$ requires $\mu^2 < 0$. This implies that

$$|\vec{v}| = \sqrt{-\frac{\mu^2}{\lambda}}$$

is the minimal field configuration.

$$\begin{aligned}
M^2_{ji} &= \left.\frac{\partial^2 V}{\partial\varphi_j \partial\varphi_i}\right|_{\vec{\varphi}=\vec{v}} \\
&= \mu^2\delta_{ij} + \lambda\left(\sum_k v_k^2\right)\delta_{ij} + 2v_j v_i \\
&= \delta_{ij}\underbrace{\left(\mu^2 + \lambda\vec{v}^2\right)}_{0} + 2\lambda v_i v_j \\
&= 2\lambda v_i v_j.
\end{aligned}$$

Choose a direction by spontaneously picking one of the vacuua:

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \qquad \Rightarrow \qquad v_i v_j = v^2 \delta_{i4}\delta_{j4},$$

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda v^2 \end{pmatrix}.$$

Looking at our symmetry groups,

$$\left.\begin{aligned} G = \mathrm{O}(4) \to n_G = 6 \text{ generators} \\ H = \mathrm{O}(3) \to n_H = 3 \text{ generators} \end{aligned}\right\} n_G - n_H = 3,$$

so we have 3 massless Goldstone bosons and one massive particle, a Higgs boson of mass[2]

$$m_H^2 = 2\lambda v^2.$$

4.2 The Higgs Mechanism

We derived the Goldstone theorem and applied it to an O(4) symmetric Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \vec{\varphi}^T \partial^\mu \vec{\varphi} - V(\vec{\varphi}), \qquad V(\vec{\varphi}) = \frac{1}{2}\mu^2|\vec{\varphi}|^2 + \frac{\lambda}{4}|\vec{\varphi}|^4.$$

$\mu^2 < 0$ implies spontaneous symmetry breaking, $\mathrm{O}(4) \to \mathrm{O}(3)$, with minimum

$$\vec{\varphi} = \vec{v}, \qquad \vec{v}^2 = \frac{-\mu^2}{\lambda}.$$

The spectrum is 1 massive state (the Higgs boson, $m_H^2 = 2\lambda\vec{v}^2$) and 3 massless states (Goldstone bosons).

Return to the complex doublet

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_3 + i\varphi_1\\ \varphi_4 + i\varphi_2\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

and define

$$\tilde{\phi} = i\tau^2\phi^* = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\phi_1^*\\ \phi_2^*\end{pmatrix} = \begin{pmatrix}\phi_2^*\\ -\phi_1^*\end{pmatrix}.$$

$\phi, \tilde{\phi}$ are $\mathrm{SU}(2)_L$ doublets with $\mathrm{U}(1)_Y$ hypercharges $y_H = \frac{1}{2}, -\frac{1}{2}$. Thus

$$D_\mu\phi = \left(\partial_\mu + igW_\mu + \tfrac{1}{2}ig'B_\mu\right)\phi,$$
$$D_\mu\tilde{\phi} = \left(\partial_\mu + igW_\mu - \tfrac{1}{2}ig'B_\mu\right)\tilde{\phi}.$$

Combine $\phi, \tilde{\phi}$ in a matrix Σ,

$$\frac{1}{\sqrt{2}}\Sigma = \left(\tilde{\phi}, \phi\right) = \begin{pmatrix}\phi_2^* & \phi_1\\ -\phi_1^* & \phi_2\end{pmatrix}, \qquad \Sigma^\dagger\Sigma = 2\phi^\dagger\phi\mathbb{1}.$$

Rewriting the Lagrangian in terms of Σ,

$$\mathscr{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) - \frac{\mu^2}{4}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2.$$

O(4) becomes $\mathrm{SU}(2)\times\mathrm{SU}(2)$,

$$\Sigma \to U_L\Sigma U_R^\dagger.$$

For the vacuum, $\varphi_4 = v \equiv |\vec{v}|$ and thus

$$\Sigma|_{\varphi_4=v} - \Sigma_0 = v\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

O(3) symmetry of $\vec{v}$ translates to SU(2) symmetry,

$$\Sigma_0 \to U\Sigma_0 U^\dagger = \Sigma_0.$$

The covariant derivative on Σ is

$$D_\mu\Sigma = \partial_\mu\Sigma + igW_\mu\Sigma - ig'\Sigma B_\mu\tfrac{\tau^3}{2}.$$

$\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y \subset \mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$, so $\mathrm{U}(1)_Y$ is the 3rd component of $\mathrm{SU}(2)_R$. Gauging the Lagrangian,

$$\mathscr{L}_{\mathrm{Higgs}}(x) = \frac{1}{4}\mathrm{tr}\left(D_\mu\Sigma^\dagger D^\mu\Sigma\right) - \frac{\mu^2}{4}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2.$$

Decomposing Σ into radial and angular components,

$$\Sigma^\dagger \Sigma = 2\phi^\dagger \phi \mathbb{1} \equiv \rho^2 \mathbb{1},$$
$$\Rightarrow \quad \Sigma = \rho \cdot U, \qquad \Sigma^\dagger \Sigma = \rho^2 U^\dagger U = \rho^2 \mathbb{1}$$

and therefore U is unitary.

We have a minimum of the action (and thus the potential) for $\rho(x) = v$. Expanding around the minimum,

$$\rho(x) = v + H(x),$$
$$\mathscr{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2}\text{tr}\left[\left(D_\mu U\right)^\dagger \left(D^\mu U\right)\right] - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

as

$$\partial_\mu \left(\rho U\right) = \left(\partial_\mu \rho\right) U + \rho \left(\partial_\mu U\right),$$
$$D_\mu \Sigma = D_\mu \left(\rho U\right) = \left(\partial_\mu \rho\right) U + \rho \left(D_\mu U\right).$$

Making the replacement $\rho = v + H$,

$$\mathscr{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{(v+H)^2}{4}\text{tr}\left[\left(D_\mu U\right)^\dagger \left(D^\mu U\right)\right] - \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{4}(v+H)^4.$$

The mass of the Higgs boson is

$$\mathscr{L} \ni H^2\left(-\frac{\mu^2}{2} - \frac{\lambda}{4}6v^2\right)$$

and using $-\mu^2 = \lambda v^2$,

$$= H^2\left(+\frac{1}{2}\lambda v^2 - \frac{3}{2}\lambda v^2\right) = -\lambda v^2 H$$
$$= -\frac{1}{2}m_H^2 H^2,$$
$$m_H^2 = 2\lambda v^2.$$

The massless Goldstone bosons are encoded in U,

$$U = \exp\left(\frac{i\chi^a \tau^a}{v}\right) \to \chi^{1,2,3} = \text{Goldstone bosons.}$$

These are massless: no derivative terms $\propto \chi^a \chi^a$ implies mass zero!

4.3 Unitary Gauge

Take a closer look at $\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right]$

$$= \mathrm{tr}\left\{\left[\partial_\mu U^\dagger - igU^\dagger W_\mu + ig' B_\mu \frac{\tau^3}{2} U^\dagger\right]\left[\partial^\mu U + igW^\mu U - ig' U B^\mu \frac{\tau^3}{2}\right]\right\}.$$

Define

$$W'_\mu = U^\dagger W_\mu U + \frac{1}{ig} U^\dagger \partial_\mu U,$$

i.e. interpret $U = U_L^\dagger$, a $\mathrm{SU}(2)_L$ gauge transformation on W_μ. We find that the trace $\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right]$

$$\begin{aligned}
&= g^2 \mathrm{tr}\left(W'_\mu W'^\mu\right) + 2\,\mathrm{tr}\left(-gg' W'_\mu B^\mu \frac{\tau^3}{2}\right) + g'^2 B_\mu B^\mu \mathrm{tr}\left(\frac{\tau^3}{2}\frac{\tau^3}{2}\right) \\
&= \frac{1}{2} g^2 W'^a_\mu W'^{\mu a} - gg' W'^3_\mu B^\mu + g'^2 \frac{1}{2} B_\mu B^\mu.
\end{aligned}$$

Taking $W_\mu \to W'_\mu$ (unitary gauge) makes U disappear and $\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right]$ gives mass terms for gauge bosons.

We have

$$\mathscr{L}_{\mathrm{Higgs}} = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{(v+H)^2}{4}\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right] - \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{4}(v+H)^4$$

and since $U = e^{i\frac{\chi^a\tau^a}{v}}$,

$$\begin{aligned}
\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) &= \mathrm{tr}\left(\partial_\mu\left(\tfrac{-i\chi^a\tau^a}{v}\right)\partial^\mu\left(\tfrac{i\chi^b\tau^b}{v}\right)\right) + \mathscr{O}\left(\chi\chi\chi\chi\right) \\
&= \frac{1}{v^2}\partial_\mu\chi^a\partial^\mu\chi^b \mathrm{tr}\left(\tau^a\tau^b\right) \\
&= \frac{2}{v^2}\partial_\mu\chi^a\partial^\mu\chi^a
\end{aligned}$$

since $\mathrm{tr}\left(\tau^a\tau^b\right) = 2\delta^{ab}$.

$$\mathscr{L} \ni \frac{v^2}{4}\frac{2}{v^2}\partial_\mu\chi^a\partial^\mu\chi^a = \frac{1}{2}\partial_\mu\chi^a\partial^\mu\chi^a.$$

$$\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right] = \frac{1}{2}g^2 W'^a_\mu W'^{\mu a} - gg' W'^3_\mu B^\mu + \frac{1}{2} g'^2 B_\mu B^\mu,$$

where $W'_\mu = U^\dagger W_\mu U + \frac{1}{ig}U^\dagger\partial_\mu U$ in the unitary gauge, so

$$\mathrm{tr}\left[\left(D_\mu U\right)^\dagger\left(D^\mu U\right)\right] = \frac{1}{2}g^2 W'^a_\mu W'^{\mu a} - gg' W'^3_\mu B^\mu + \frac{1}{2} g'^2 B_\mu B^\mu$$

$$= \frac{1}{2} g^2 \left(W'^{1}_{\mu} W'^{\mu 1} + W'^{2}_{\mu} W'^{\mu 2} \right) + \underbrace{\frac{1}{2} \left(W'^{3}_{\mu} B_{\mu} \right) \underbrace{\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}}_{G} \begin{pmatrix} W'^{\mu 3} \\ B^{\mu} \end{pmatrix}}_{T^3}.$$

We want to find the eigenvalues of G.

$$\mathrm{tr}(G) = g^2 + g'^2,$$
$$\det(G) = g^2 g'^2 - \left(gg'\right)^2 = 0.$$

Recall that

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}},$$
$$\begin{pmatrix} W'^{3}_{\mu} \\ B_{\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_{\mu} \\ A_{\mu} \end{pmatrix}.$$

Now,

$$\begin{aligned}
\left(g^2 + g'^2\right) T_3 &= \left(Z_\mu, A_\mu\right) \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix}^T \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \begin{pmatrix} Z^{\mu} \\ A^{\mu} \end{pmatrix} \\
&= \left(Z_\mu, A_\mu\right) \begin{pmatrix} g & -g' \\ g' & g \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \begin{pmatrix} Z^{\mu} \\ A^{\mu} \end{pmatrix} \\
&= \left(Z_\mu, A_\mu\right) \begin{pmatrix} g^3 + gg'^2 & -g^2 g' - g'^3 \\ g' g^2 - g^2 g' & -gg'^2 + gg'^2 \end{pmatrix} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \begin{pmatrix} Z^{\mu} \\ A^{\mu} \end{pmatrix} \\
&= \left(Z_\mu, A_\mu\right) \begin{pmatrix} g^4 + g^2 g'^2 + g^2 g'^2 + g'^4 & g^3 g' + gg'^3 - g^3 g' - g'^3 g \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^{\mu} \\ A^{\mu} \end{pmatrix} \\
&= \left(Z_\mu, A_\mu\right) \begin{pmatrix} \left(g^2 + g'^2\right)^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^{\mu} \\ A^{\mu} \end{pmatrix}.
\end{aligned}$$

So,

$$\mathscr{L}_{\text{Higgs}} \ni m_W^2 W^{+}_{\mu} W^{\mu}_{-} + \frac{1}{2} m_Z^2 Z_{\mu} Z^{\mu},$$
$$m_W^2 = \frac{g^2}{4} v^2, \qquad m_Z^2 = \frac{g^2 + g'^2}{4} v^2, \qquad m_\gamma^2 = 0.$$

The Lagrangian is thus

$$\mathscr{L}_{\text{Higgs}} = \frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{2} m_H^2 H^2 + \left(1 + 2\frac{H}{v} + \frac{H^2}{v^2}\right) \left(m_W^2 W'^{+}_{\mu} W'^{\mu}_{-} + \frac{1}{2} m_Z^2 Z'_{\mu} Z'^{\mu} \right) - \frac{3\lambda}{4} v H^3 - \frac{\lambda}{4} H^4.$$

This gives us:

- the Higgs boson of mass m_H and self couplings $\sim H^3$, H^4,
- mass terms for $W^{\pm}$, Z bosons,
- interactions $HW^{+}_{\mu} W^{-\mu}$, $HZ_{\mu} Z^{\mu}$, $H^2 W^{+}_{\mu} W^{-\mu}$, $H^2 Z_{\mu} Z^{\mu}$.

5 Resumé

1. The gauge symmetry $SU(2)_L \times U(1)_Y$ is still present at the level of the Lagrangian; it is only hidden by expanding fields about the minimum field configuration or vacuum state.

2. The 3 Goldstone bosons are eaten by the $W^\pm$, Z-bosons, which play the role of longditudinal degrees of freedom.

5.1 Remark on SSB

In QFT, SSB is signalled by a non-zero vacuum expectation value (VEV). In the Higgs doublet, we would have something like

$$\langle 0|\tilde{\phi}|0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

which is equivalent to a field configuration minimising the potential.

As an analogy, consider SSB in 2 flavour massless QCD.

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{u}, \bar{d}\right),$$

$$\mathscr{L} = \bar{\psi} \not{D} \psi = \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R.$$

Under $SU(2)_L \times SU(2)_R$ symmetry, $\psi_{\substack{L\\R}} \to U_{\substack{L\\R}} \psi_{\substack{L\\R}}$. There is a vacuum expectation value of

$$\langle 0|\bar{\psi}\psi|0\rangle \neq 0.$$

$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$ breaks the $SU(2)_L \times SU(2)_R$ symmetry, but

$$\bar{\psi}\psi \to \bar{\psi} U^\dagger U \psi = \bar{\psi}\psi.$$

Therefore under SSB, $SU(2)_L \times SU(2)_R \to SU(2)_V$ as in the Higgs mechanism and we get 3 massless Goldstone bosons. In QCD, the lightest particles are the πs: $\pi^\pm, \pi^0$ ($m_\pi \approx 140$ MeV). The next heaviest is ρ, $m_\rho \approx 770$ MeV. Identify the Goldstone bosons with πs (pseudo-Goldstone bosons) and add quark masses as a perturbation. Thus $m_\pi \propto (m_u + m_d)$ are generated.

We could think of the Higgs boson as a composite particle. Could SSB in QCD generate m_W, m_Z? Yes! However, the scale is much too low.

$$\left.\begin{aligned} m_W &\approx 80 \text{ GeV} \\ m_Z &\approx 90 \text{ GeV} \end{aligned}\right\} \text{ set by } v = 250 \text{ GeV (Fermi scale).}$$

In QCD, the corresponding scale is $f_\pi \approx 93$ MeV. This means m_W, m_Z generated by SSB in QCD would be too small by a factor $\frac{f_\pi}{v} = \mathcal{O}(10^{-3})$.

The basic idea of technicolour models is to introduce an additional "charge" technicolour. The most naïve model simply rescales QCD by $\frac{v}{f_\pi}$. The simplest idea is in conflict with data from experiments, but dynamical SSB of electroweak symmetry is still very attractive.

6 The Yukawa Lagrangian

Our goal is to obtain masses for the fermions without breaking the gauge symmetry. The idea is to couple the Higgs doublet ϕ to the fermions in $\mathscr{L}_{\text{Yukawa}}$.

For the first lepton family,

$$\mathscr{L}_{\text{Yukawa}} = -c_e \bar{\psi}_{e,R} \phi^\dagger \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L - c_e \left(\bar{\psi}_{\nu_e}, \psi_e \right)_R \phi \psi_{e,R}.$$

Under SSB,

$$\phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_3 + i\varphi_1 \\ v + H + i\varphi_2 \end{pmatrix} \xrightarrow[\text{gauge}]{\text{unitary}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}$$

and

$$\begin{aligned} \mathscr{L}_{\text{Yukawa}} &\ni -c_e \left(\frac{v}{\sqrt{2}} \bar{\psi}_{e,R} \psi_{e,L} + \frac{v}{\sqrt{2}} \bar{\psi}_{e,L} \psi_{e,R} \right) \\ &= -c_e \frac{v}{\sqrt{2}} \bar{\psi}_e \psi_e = -m_e \bar{\psi}_e \psi_e. \end{aligned}$$

Proceed analogously for all fermions.

Combining all fields into a multiplet ψ,

$$\psi = \begin{pmatrix} \psi_{\nu_e,L} \\ \psi_{e,L} \\ \vdots \\ \psi_{b,R} \end{pmatrix}, \qquad D_\mu \psi = \left(\partial_\mu + i g W_\mu^a T^a + i g' B_\mu Y \right) \psi.$$

For the interaction of fermions with $W_\mu^\pm$, Z, γ,

$$\bar{\psi} i \not{D} \psi = \bar{\psi} i \not{\partial} \psi \underbrace{- \bar{\psi} \gamma^\mu \left(g W_\mu^a T^a + g' B_\mu Y \right) \psi}_{\mathscr{L}_{\text{int}}}.$$

Passing to $W^\pm$, Z, γ,

$$\mathscr{L}_{\text{int}} = -e \left\{ A_\mu J^\mu_{\text{EM}} + \frac{1}{\sin\theta_W \cos\theta_W} Z_\mu J^\mu_{\text{NC}} + \frac{1}{\sqrt{2} \sin\theta_W} \left(W_\mu^+ J^\mu_{\text{cc}} + W_\mu^- J^\mu_{\text{cc}} \right) \right\}$$

where

$$J^\mu_{\text{EM}} = \bar{\psi}\gamma^\mu Q\psi, \qquad J^\mu_{\text{NC}} = \bar{\psi}\gamma^\mu \left[T^3 - \sin^2\theta_W Q\right]\psi$$
$$= \bar{\psi}\gamma^\mu T^3\psi - \sin^2\theta_W J^\mu_{\text{EM}},$$

and the charge current is

$$J^\mu_{\text{cc}} = \bar{\psi}\gamma^\mu \left(T^1 + iT^2\right)\psi.$$

6.1 Fermion Mass Matrix

We need to form $\mathrm{SU}(2)_L$ singlets by combining the Higgs doublet ϕ with fermion doublets $\binom{\psi_1}{\psi_2}_L$. This can be done by

$$\phi^\dagger \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_L, \qquad \tilde{\phi}^\dagger \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_L,$$

using $\tilde{\phi} = i\tau^2\phi^* = \binom{\phi_2^*}{-\phi_1^*}$, and the Hermitian conjugates

$$\left(\bar{\psi}_1, \bar{\psi}_2\right)_L \phi, \qquad \left(\bar{\psi}_1, \bar{\psi}_2\right)_L \tilde{\phi}.$$

The most general Lagrangian is

$$\mathscr{L}_{\text{Yuk}} = -\left(\bar{\psi}_{e,R}, \bar{\psi}_{\mu,R}, \bar{\psi}_{\tau,R}\right) c_l \begin{pmatrix} \phi^\dagger \begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix}\psi_{\nu_\mu}\\ \psi_\mu\end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix}\psi_{\nu_\tau}\\ \psi_\tau\end{pmatrix}_L \end{pmatrix} + \text{h.c.}$$
$$- \left(\bar{\psi}_{u,R}, \bar{\psi}_{c,R}, \bar{\psi}_{t,R}\right) c'_q \begin{pmatrix} \tilde{\phi}^\dagger \begin{pmatrix}\psi_u\\ \psi_{d'}\end{pmatrix}_L \\ \tilde{\phi}^\dagger \begin{pmatrix}\psi_c\\ \psi_{s'}\end{pmatrix}_L \\ \tilde{\phi}^\dagger \begin{pmatrix}\psi_t\\ \psi_{b'}\end{pmatrix}_L \end{pmatrix} + \text{h.c.}$$
$$- \left(\bar{\psi}_{d',R}, \bar{\psi}_{s',R}, \bar{\psi}_{b',R}\right) c_q \begin{pmatrix} \phi^\dagger \begin{pmatrix}\psi_u\\ \psi_{d'}\end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix}\psi_c\\ \psi_{s'}\end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix}\psi_t\\ \psi_{b'}\end{pmatrix}_L \end{pmatrix} + \text{h.c.}$$

This is the full Yukawa Lagrangian, assuming neutrino masses are zero.

- c_l, c'_q, c_q are *a priori* arbitrary complex 3×3 matrices.
- We have $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ symmetry.
- Primed fields $\psi_{d'}, \psi_{s'}, \psi_{b'}$ allow for a mismatch between
    1. fields coupling to $W^\pm_\mu, Z$ bosons,
    2. eigenstates of the mass matrix.

We can change the basis of fields with the same quantum numbers, e.g.

$$\begin{pmatrix} \psi_{e,R} \\ \psi_{\mu,R} \\ \psi_{\tau,R} \end{pmatrix} \to U_1 \begin{pmatrix} \psi_{e,R} \\ \psi_{\mu,R} \\ \psi_{\tau,R} \end{pmatrix}, \quad U_1 \in \mathrm{U}(3).$$

Therefore $\mathscr{L}$ is unchanged except for $\mathscr{L}_{\text{Yuk}}$,

$$c_l \to U_1^\dagger c_l.$$

Similar changes can be made for left-handed doublets.

$$\begin{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L \\ \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}_L \\ \begin{pmatrix} \psi_t \\ \psi_b \end{pmatrix}_L \end{pmatrix} \to V_2 \otimes \mathbb{1} \begin{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L \\ \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}_L \\ \begin{pmatrix} \psi_t \\ \psi_b \end{pmatrix}_L \end{pmatrix}.$$

Using the freedom to change basis, we see

$$\begin{aligned} c_l &\to U_1^\dagger c_l V_1, \\ c'_q &\to U_2^\dagger c'_q V_2, \\ c_q &\to U_3^\dagger c_q V_2. \end{aligned}$$

For leptons,

$$c_l c_l^\dagger \to U_1^\dagger c_l c_l^\dagger U_1.$$

Choose U_1 such that $c_l c_l^\dagger$ is diagonal.

$$c_l c_l^\dagger = \begin{pmatrix} c_e^2 & 0 & 0 \\ 0 & c_\mu^2 & 0 \\ 0 & 0 & c_\tau^2 \end{pmatrix} \quad \Rightarrow \quad c_l = \begin{pmatrix} c_e & 0 & 0 \\ 0 & c_\mu & 0 \\ 0 & 0 & c_\tau \end{pmatrix} W$$

where W is arbitrary. Choose $V_1 = W^\dagger$. Then

$$c_l = \begin{pmatrix} c_e & 0 & 0 \\ 0 & c_\mu & 0 \\ 0 & 0 & c_\tau \end{pmatrix} \geq 0.$$

It is the same for quarks:

$$c_q' = \begin{pmatrix} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{pmatrix} \geq 0.$$

$$c_q c_q^\dagger \to U_3^\dagger c_q c_q^\dagger U_3 \quad \Rightarrow \quad c_q = \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V^\dagger.$$

s cannot be reduced further as V_2 has been used up for c_q'. The standard form for c_q is to use the U_3 transformation to obtain

$$c_q = V \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V^\dagger, \quad V = V_{\text{CKM}}.$$

$\mathscr{L}_{\text{Yuk}}$ in the unitary gauge is

$$\begin{aligned} \mathscr{L}_{\text{Yuk}} = \Bigg\{ &- \left(\bar{\psi}_{e,R}, \bar{\psi}_{\mu,R}, \bar{\psi}_{\tau,R}\right) \begin{pmatrix} c_e & 0 & 0 \\ 0 & c_\mu & 0 \\ 0 & 0 & c_\tau \end{pmatrix} \begin{pmatrix} \psi_{e,L} \\ \psi_{\mu,L} \\ \psi_{\tau,L} \end{pmatrix} + \text{h.c.} \\ &- \left(\bar{\psi}_{u,R}, \bar{\psi}_{c,R}, \bar{\psi}_{t,R}\right) \begin{pmatrix} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{pmatrix} \begin{pmatrix} \psi_{u,L} \\ \psi_{c,L} \\ \psi_{t,L} \end{pmatrix} + \text{h.c.} \\ &- \left(\bar{\psi}_{d',R}, \bar{\psi}_{s',R}, \bar{\psi}_{b',R}\right) V_{\text{CKM}} \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V_{\text{CKM}}^\dagger \begin{pmatrix} \psi_{d',L} \\ \psi_{s',L} \\ \psi_{b',L} \end{pmatrix} + \text{h.c.} \Bigg\} \frac{v}{\sqrt{2}} \left(1 + \frac{H}{v}\right) \end{aligned}$$

We can now get the masses:

$$m_e = c_e \frac{v}{\sqrt{2}}, \quad \ldots, \quad m_b = c_b \frac{v}{\sqrt{2}}.$$

The Higgs coupling is proportional to the fermion mass. We have mass eigenstates

$$\begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix} = V_{\text{CKM}}^\dagger \begin{pmatrix} \psi_d' \\ \psi_s' \\ \psi_b' \end{pmatrix}.$$