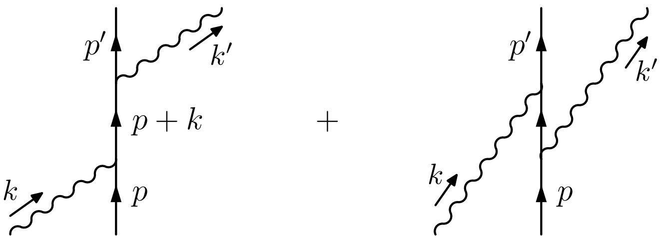
# QED at Tree Level

We'll look at the Compton scattering process ($e^- \gamma \to e^- \gamma$).




Using $\epsilon_\nu(k), \epsilon^*_\mu(k')$ for initial and final photon polarisation vectors, Feynman rules give us the Feynman amplitude (to tree level):

$$\begin{aligned}
i\mathcal{M} = \bar{u}(p')&(-ie\gamma^\mu)\epsilon^*_\mu(k')\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^\nu)\epsilon_\nu(k)u(p) \\
&+\bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(-ie\gamma^\mu)\epsilon^*_\mu(k')u(p) \\
= & -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p).
\end{aligned}$$

Since $p^2 = m^2$ and $k^2 = k'^2 = 0$,

$$(p+k)^2 - m^2 = 2p\cdot k, \qquad (p-k')^2 - m^2 = -2p\cdot k'.$$

The numerators simplify via Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, since

$$\begin{aligned}
(\not{p}+m)\gamma^\nu u(p) &= (\gamma^\mu p_\mu + m)\gamma^\nu u(p) \\
&= (p_\mu\gamma^\mu\gamma^\nu + m\gamma^\nu)u(p) \\
&= (p_\mu\{2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu\} + m\gamma^\nu)u(p) \\
&= (2p^\nu - \gamma^\nu\not{p} + m\gamma^\nu)u(p) \\
&= 2p^\nu u(p) - \gamma^\nu(\not{p}-m)u(p) \\
&= 2p^\nu u(p)
\end{aligned}$$

as

$$\gamma^\nu(\not{p} - m)u(p) = \gamma^\nu(i\not{\partial} - m)u(p)$$

and $(i\gamma^\mu\partial_\mu - m)\psi = 0$ is the Dirac equation.

This gives us

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p)$$

for the Feynman amplitude.

For any QED process with an external photon, we define the amplitude

$$i\mathcal{M} \equiv i\mathcal{M}^\mu(k)\epsilon^*_\mu(k).$$

Then the cross section is proportional to

$$\sum_\epsilon |\epsilon^*_\mu(k)\mathcal{M}^\mu(k)|^2 = \sum_\epsilon \epsilon^*_\mu\epsilon_\nu\mathcal{M}^\mu\mathcal{M}^{\nu *}.$$

If we orient $k$ in the $z$-direction, $k^\mu = (k, 0, 0, k)$, we can choose our two polarisation vectors to be

$$\epsilon^\mu_1 = (0, 1, 0, 0), \qquad \epsilon^\mu_2 = (0, 0, 1, 0).$$

Then

$$\sum_\epsilon |\epsilon^*_\mu(k)\mathcal{M}^\mu(k)|^2 = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2.$$

External photons are created by the interaction term

$$\int \mathrm{d}^4x\, ej^\mu A_\mu, \qquad j^\mu = \bar{\psi}\gamma^\mu\psi,$$

so we expect $\mathcal{M}^\mu(k)$ to be given by a matrix element of the Heisenberg field $j^\mu$:

$$\mathcal{M}^\mu(k) = \int \mathrm{d}^4x\, e^{ik\cdot x}\langle f|j^\mu(x)|i\rangle\,,$$

where the initial and final states $|i\rangle$, $|f\rangle$ include all particles except the external photon.

If $\partial_\mu j^\mu = 0$ remains valid on a quantum level, then

$$k_\mu\mathcal{M}^\mu(k) = k_\mu\int \mathrm{d}^4x\, e^{ik\cdot x}\langle f|j^\mu(x)|i\rangle = 0.$$

This is the Ward identity.

There is a risqué physical explanation also, if the argument involving $\partial_\mu j^\mu = 0$ remains elusive. We have a non-zero cross section for the set of all possible polarisation vectors, the two-dimensional polarisation subspace spanned by $\epsilon^\mu_1, \epsilon^\mu_2$. If we now choose a polarisation vector that is outside this subspace, the cross section will obviously vanish. This is what we have done, since $k^\mu$ is such a vector.

⋒ Corrections to `fionnf@maths.tcd.ie`.