Supersymmetry

A simple supersymmetric theory is described by the Lagrangian

$$\mathcal{L} = \underbrace{\partial_\mu \phi^* \partial^\mu \phi}_{\mathcal{L}_1} + \underbrace{i\chi^\dagger \bar{\sigma} \cdot \partial \chi}_{\mathcal{L}_2} + \underbrace{F^* F}_{\mathcal{L}_3}, \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

for ϕ a complex scalar field (bosonic), χ a left-handed Weyl spinor (fermionic) and F an auxiliary complex scalar field. This Lagrangian is invariant under the infinitesimal (supersymmetric) transformations (up to $\mathcal{O}(\epsilon)$)

$$\delta\phi = -i\epsilon^T \sigma^2 \chi, \tag{1}$$

$$\delta\chi = \epsilon F + \sigma \cdot \partial\phi\, \sigma^2 \epsilon^*, \tag{2}$$

$$\delta F = -i\epsilon^\dagger \bar{\sigma} \cdot \partial \chi, \tag{3}$$

Note the sign in eq. 2.

where $\epsilon = \left(\begin{smallmatrix} \epsilon_1 \\ \epsilon_2 \end{smallmatrix}\right)$ is a 2-component spinor of Grassmann numbers, so it anti-commutes with itself and other fermions.

To tackle this, we must find the conjugates of the transformations of the various fields: $\delta\phi^*, \delta\chi^\dagger, \delta F^*$. The following identities will be useful. Recall that σ is self-adjoint ($\sigma = \sigma^\dagger$) and

$$\sigma^\mu = (\mathbb{1}, \sigma^a), \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^a).$$

Then (for $a = 1, 2, 3$), by the Grassmann property $(\alpha\beta)^* = \beta^* \alpha^*$,

$$\sigma^2 \sigma^a \sigma^2 = -(\sigma^a)^*, \quad \sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^*,$$
$$\chi^T \epsilon = \chi^a \epsilon^a = -\epsilon^a \chi^a = -\epsilon^T \chi,$$
$$\chi^T \sigma^2 \epsilon = \chi^a \sigma^2_{ab} \epsilon^b = -\chi^a \sigma^2_{ba} \epsilon^b = \epsilon^T \sigma^2 \chi.$$

Now, $\delta\phi$ under * becomes

$$\begin{aligned} \delta\phi^* &= (-i\epsilon^T \sigma^2 \chi)^* \\ &= i(\epsilon^T \sigma^2 \chi)^* \\ &= i(\epsilon^a \sigma^2_{ab} \chi^b)^* \\ &= i(\chi^b)^* (\sigma^2_{ab})^* (\epsilon^a)^* \\ &= i\chi^\dagger \sigma^2 \epsilon^*. \end{aligned}$$

$\delta\chi$ under † becomes

$$\begin{aligned}\delta\chi^\dagger &= (\epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*)^\dagger \\ &= \epsilon^\dagger F^* + \epsilon^{*\dagger}(\sigma^2)^\dagger(\sigma\cdot\partial\phi)^\dagger \\ &= \epsilon^\dagger F^* + \epsilon^T\sigma^2(\sigma^\mu\partial_\mu\phi)^\dagger \\ &= \epsilon^\dagger F^* + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu.\end{aligned}$$

δF under * becomes

$$\begin{aligned}\delta F^* &= (-i\epsilon^\dagger\bar\sigma\cdot\partial\chi)^* \\ &= i(\epsilon^\dagger\bar\sigma\cdot\partial\chi)^* \\ &= i(\bar\sigma^\mu\partial_\mu\chi)^*\epsilon^T \\ &= i(\partial_\mu\chi)^*(\bar\sigma^\mu)^\dagger\epsilon^T \\ &= i\partial_\mu\chi^\dagger\bar\sigma^\mu\epsilon^T.\end{aligned}$$

We can now study the transformation of the Lagrangian. As $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ and $\delta\mathcal{L} = \delta\mathcal{L}_1 + \delta\mathcal{L}_2 + \delta\mathcal{L}_3$, we will do this term-wise.

$$\begin{aligned}\delta\mathcal{L}_1 &= \delta\Big[\partial_\mu\phi^*\partial^\mu\phi\Big] \\ &= \partial_\mu(\delta\phi^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(\delta\phi) \\ &= \partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi).\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L}_2 &= \delta\Big[i\chi^\dagger\bar\sigma\cdot\partial\chi\Big] \\ &= i(\delta\chi^\dagger)\bar\sigma\cdot\partial\chi + i\chi^\dagger\bar\sigma\cdot\partial(\delta\chi) \\ &= i(\epsilon^\dagger F^* + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\bar\sigma\cdot\partial\chi + i\chi^\dagger\bar\sigma\cdot\partial(\epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*) \\ &= i\epsilon^\dagger F^*\bar\sigma\cdot\partial\chi + i\epsilon^T\sigma^2\underbrace{\partial_\mu\phi^*\sigma^\mu\bar\sigma\cdot\partial\chi}_{\partial_\mu\phi^*\partial^\mu\chi} \\ &\quad + i\chi^\dagger\bar\sigma\cdot\partial(\epsilon F) + i\chi^\dagger\underbrace{\bar\sigma\cdot\partial(\sigma\cdot\partial\phi}_{\partial_\mu\partial^\mu\phi}\,\sigma^2\epsilon^*)\end{aligned}$$

and noting that $\partial_\mu\phi^*\sigma^\mu\bar\sigma^\nu\partial_\nu\chi = \partial_\mu\phi^*\partial^\mu\chi$ and $\bar\sigma^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi) = \partial_\mu\partial^\mu\phi$,

$$\begin{aligned}&= i\epsilon^\dagger F^*\bar\sigma\cdot\partial\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\partial^\mu\chi \\ &\quad + i\chi^\dagger\bar\sigma\cdot\partial(\epsilon F) + i\chi^\dagger\partial_\mu\partial^\mu\phi\,\sigma^2\epsilon^*\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L}_3 &= \delta\Big[F^*F\Big] \\ &=(\delta F^*)F + F^*(\delta F) \\ &=i\partial_\mu\chi^\dagger\bar\sigma^\mu\epsilon^T F + F^*(-i\epsilon^\dagger\bar\sigma\cdot\partial\chi).\end{aligned}$$

Putting this together, our transformed Lagrangian is

$$\begin{aligned}\delta\mathcal{L} = &\underbrace{\partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi}_{(1)} + \underbrace{\partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi)}_{(2)} \\ &+ \underbrace{i\epsilon^\dagger F^*\bar{\sigma}\cdot\partial\chi}_{(3)} + \underbrace{i\epsilon^T\sigma^2\partial_\mu\phi^*\partial^\mu\chi}_{(4)} \\ &+ \underbrace{i\chi^\dagger\bar{\sigma}\cdot\partial(\epsilon F)}_{(5)} + \underbrace{i\chi^\dagger\partial_\mu\partial^\mu\phi\,\sigma^2\epsilon^*}_{(6)} \\ &+ \underbrace{i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon^T F}_{(7)} + \underbrace{F^*(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi)}_{(8)}.\end{aligned}$$

Noting that

$$(2)+(4)=0, \quad (3)+(8)=0$$

leaves

$$\begin{aligned}\delta\mathcal{L} = &\underbrace{\partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi}_{(1)} \\ &+ \underbrace{i\chi^\dagger\bar{\sigma}\cdot\partial(\epsilon F)}_{(5)} + \underbrace{i\chi^\dagger\partial_\mu\partial^\mu\phi\,\sigma^2\epsilon^*}_{(6)} \\ &+ \underbrace{i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon^T F}_{(7)}.\end{aligned}$$

Since $\chi^\dagger \to 0$ at ∞, integrating (6) by parts gives

$$\begin{aligned}\int \mathrm{d}^4x\, i\chi^\dagger\partial_\mu\partial^\mu\phi\,\sigma^2\epsilon^* &= 0 - \int \mathrm{d}^4x\, i\partial_\mu\chi^\dagger\partial^\mu\phi\,\sigma^2\epsilon^*, \\ i\chi^\dagger\partial_\mu\partial^\mu\phi\,\sigma^2\epsilon^* &= -i\partial_\mu\chi^\dagger\partial^\mu\phi\,\sigma^2\epsilon^*\end{aligned}$$

so $(6)+(1)=0$. Similarly, $(5)+(7)=0$ as

$$\int \mathrm{d}^4x\, i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon^T F = 0 - \int \mathrm{d}^4x\, i\chi^\dagger\bar{\sigma}^\mu\epsilon^T\partial_\mu F.$$

$\delta\mathcal{L}=0$ up to a total divergence and hence is invariant under the supersymmetry transformation.

We could also introduce a supersymmetric mass term

$$\mathcal{L}_4 = \left[m\phi F + \frac{i}{2}m\chi^T\sigma^2\chi\right] + \text{c.c.},$$

which we will show to be invariant.

$$\begin{aligned}\delta\mathcal{L}_4 &= \delta\left[m\phi F + \frac{i}{2}m\chi^T\sigma^2\chi\right] + \text{c.c.}\\ &= \left[m(\delta\phi)F + m\phi(\delta F) + \frac{i}{2}m(\delta\chi^T)\sigma^2\chi + \frac{i}{2}m\chi^T\sigma^2(\delta\chi)\right] + \text{c.c.}\end{aligned}$$

Using the Grassmann properties of χ,

$$\begin{aligned}(\delta\chi^T)\sigma^2\chi &= (\delta\chi)^a\sigma^2_{ab}\chi^b\\ &= -(\delta\chi)^a\sigma^2_{ba}\chi^b\\ &= \chi^T\sigma^2(\delta\chi)\end{aligned}$$

and inserting $\delta\chi = \epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*$,

$$\begin{aligned}\chi^T\sigma^2(\delta\chi) &= \chi^T\sigma^2(\epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*)\\ &= \chi^T\sigma^2\epsilon F + \chi^T\sigma^2\sigma\cdot\partial\phi\,\sigma^2\epsilon^*.\end{aligned}$$

Examining this second term,

$$\begin{aligned}\chi^T\sigma^2\sigma\cdot\partial\phi\,\sigma^2\epsilon^* &= \chi^T\sigma^2\sigma^\mu\partial_\mu\phi\,\sigma^2\epsilon^*\\ &= \chi^T\partial_\mu\phi\,\sigma^2\sigma^\mu\sigma^2\epsilon^*\\ &= \chi^T\partial_\mu\phi\,(\bar{\sigma}^\mu)^*\epsilon^*\\ &= \partial_\mu\phi\,\underbrace{\chi^T(\bar{\sigma}^\mu)^*\epsilon^*}_{\maltese}.\end{aligned}$$

$$\begin{aligned}\maltese &= \chi^T(\bar{\sigma}^\mu)^*\epsilon^*\\ &= \chi^a(\bar{\sigma}^\mu)^*_{ab}(\epsilon^*)^b\\ &= -(\epsilon^*)^b(\bar{\sigma}^\mu)^*_{ab}\chi^a\\ &= -(\epsilon^*)^b(\bar{\sigma}^\mu)^\dagger_{ba}\chi^a\\ &= -\epsilon^\dagger\bar{\sigma}^\mu\chi.\end{aligned}$$

We now know that

$$\begin{aligned}\chi^T\sigma^2(\delta\chi) &= \chi^T\sigma^2\epsilon F + \chi^T\sigma^2\sigma\cdot\partial\phi\,\sigma^2\epsilon^*\\ &= \chi^T\sigma^2\epsilon F + \chi^T\sigma^2\sigma\cdot\partial\phi\,\sigma^2\epsilon^*\\ &= \chi^T\sigma^2\epsilon F + \partial_\mu\phi\,\chi^T(\bar{\sigma}^\mu)^*\epsilon^*\\ &= \chi^T\sigma^2\epsilon F - \partial_\mu\phi\,\epsilon^\dagger\bar{\sigma}^\mu\chi,\end{aligned}$$

so the Lagrangian reads

$$\begin{aligned}\delta\mathcal{L}_4 &= \Big[m(\delta\phi)F + m\phi(\delta F) + im\chi^T\sigma^2(\delta\chi)\Big] + \text{c.c.}\\ &= \Big[m(-i\epsilon^T\sigma^2\chi)F + m\phi(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi)\\ &\qquad + im\chi^T\sigma^2\epsilon F - im\partial_\mu\phi\,\epsilon^\dagger\bar{\sigma}^\mu\chi \qquad \Big] + \text{c.c.}\end{aligned}$$

and noting that $\phi \to 0$ at ∞,

$$\begin{aligned}\int \mathrm{d}^4x\, im\partial_\mu\phi\,\epsilon^\dagger\bar{\sigma}^\mu\chi &= 0 - \int \mathrm{d}^4x\, im\phi\,\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi\\ im\partial_\mu\phi\,\epsilon^\dagger\bar{\sigma}^\mu\chi &= -im\phi\,\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi.\end{aligned}$$

This leaves us with

$$\begin{aligned}\delta\mathcal{L}_4 &= \Big[m(-i\epsilon^T\sigma^2\chi)F + im\chi^T\sigma^2\epsilon F\Big] + \text{c.c.}\\ &= 0,\end{aligned}$$

so the Lagrangian $\mathcal{L} + \mathcal{L}_4$ describes a massive supersymmetric quantum field theory.

However, there still remains the auxiliary scalar field F. We can remove this by first rewriting $\mathcal{L}_4$ as

$$\begin{aligned}\mathcal{L}_4 &= \left[m\phi F + \frac{i}{2}m\chi^T\sigma^2\chi\right] + \text{c.c.}\\ &= m\left[\phi F + \phi^* F^* + \frac{i}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]\end{aligned}$$

as $(i\chi^T\sigma^2\chi)^* = -i\chi^\dagger\sigma^2\chi^*$. Then, by Euler–Lagrange,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu F)} - \frac{\partial\mathcal{L}}{\partial F} = 0 \quad\Rightarrow\quad F^* = -m\phi.$$

By substitution,

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi + i\chi^\dagger\bar{\sigma}\cdot\partial\chi + F^*F + m\left[\phi F + \phi^* F^* + \frac{i}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]\\ &= \partial_\mu\phi^*\partial^\mu\phi - m^2\phi\phi^* + i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right).\end{aligned}$$

⬡ Corrections to `fionnf@maths.tcd.ie`.