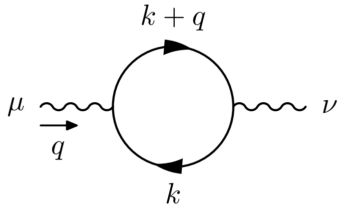
Dimensional Regularisation of the Photon Self-Energy

The photon self-energy is a modification to the photon structure by a virtual electron-positron pair.



In d-dimensional spacetime with $d \in \mathbb{C}$ and Minkowski metric η, we can define the Dirac matrices as operators generating the Clifford algebra $\mathcal{C}_{1,d-1}(\mathbb{R})$ using the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, with $\operatorname{tr}(\mathbb{1}) \equiv 4$ for $d \sim 4$.

For a closed loop of fermion propagators we get a fermion loop factor of -1 and a trace of Dirac matrices. Feynman rules give us

$$\begin{aligned} i\mathcal{M} = i\Pi_2^{\mu\nu}(q) &= (-ie)^2(-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\operatorname{tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \\ &= -e^2\int \frac{\mathrm{d}^d k}{(2\pi)^d}\operatorname{tr}\left[\gamma^\mu \frac{\not{k}+m}{k^2-m^2}\gamma^\nu \frac{\not{k}+\not{q}+m}{(k+q)^2-m^2}\right], \end{aligned}$$

the second-order contribution to the photon polarisation tensor.

Using our trace technology,

$$\begin{aligned} &\operatorname{tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right] \\ &\qquad = \operatorname{tr}\left[\gamma^\mu(\gamma^\rho k_\rho + m)\gamma^\nu(\gamma^\sigma k_\sigma + \gamma^\sigma q_\sigma + m)\right] \\ &\qquad = \operatorname{tr}\left[(\gamma^\mu\gamma^\rho k_\rho\gamma^\nu + \gamma^\mu m\gamma^\nu)(\gamma^\sigma k_\sigma + \gamma^\sigma q_\sigma + m)\right] \\ &\qquad = \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma k_\sigma + \gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma q_\sigma + \gamma^\mu\gamma^\rho k_\rho\gamma^\nu m \right. \\ &\qquad\qquad \left. + \gamma^\mu m\gamma^\nu\gamma^\sigma k_\sigma + \gamma^\mu m\gamma^\nu\gamma^\sigma q_\sigma + \gamma^\mu m\gamma^\nu m\right] \\ &\qquad = \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma k_\sigma\right] + \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma q_\sigma\right] + \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu m\right] \\ &\qquad\quad + \operatorname{tr}\left[\gamma^\mu m\gamma^\nu\gamma^\sigma k_\sigma\right] + \operatorname{tr}\left[\gamma^\mu m\gamma^\nu\gamma^\sigma q_\sigma\right] + \operatorname{tr}\left[\gamma^\mu m\gamma^\nu m\right] \\ &\qquad = \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma k_\sigma\right] + \operatorname{tr}\left[\gamma^\mu\gamma^\rho k_\rho\gamma^\nu\gamma^\sigma q_\sigma\right] + \operatorname{tr}\left[\gamma^\mu m\gamma^\nu m\right] \end{aligned}$$

as the trace of an odd number of gamma matrices is zero. Then

$$\begin{aligned}\mathrm{tr}\left[\gamma^\mu \gamma^\rho k_\rho \gamma^\nu \gamma^\sigma k_\sigma\right] &= \mathrm{tr}\left[\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma k_\rho k_\sigma\right] \\ &= 4\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}\right)\left(k_\rho k_\sigma\right) \\ &= 4\left(k^\mu k^\nu - \eta^{\mu\nu} k^\sigma k_\sigma + k^\nu k^\mu\right) \\ &= 4\left(2k^\mu k^\nu - \eta^{\mu\nu} k^\sigma k_\sigma\right),\end{aligned}$$

$$\begin{aligned}\mathrm{tr}\left[\gamma^\mu \gamma^\rho k_\rho \gamma^\nu \gamma^\sigma q_\sigma\right] &= \mathrm{tr}\left[\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma k_\rho q_\sigma\right] \\ &= 4\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}\right)\left(k_\rho q_\sigma\right) \\ &= 4\left(k^\mu q^\nu - \eta^{\mu\nu} k^\sigma q_\sigma + k^\nu q^\mu\right),\end{aligned}$$

and

$$\mathrm{tr}\left[\gamma^\mu m \gamma^\nu m\right] = 4\eta^{\mu\nu} m^2,$$

so

$$\mathrm{tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right] = 4\left(2k^\mu k^\nu + k^\mu q^\nu + \eta^{\mu\nu}(m^2 - k^2 - k^\sigma q_\sigma)\right).$$

Returning to our amplitude,

$$i\Pi_2^{\mu\nu}(q) = -\,4e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2k^\mu k^\nu + k^\mu q^\nu + \eta^{\mu\nu}(m^2 - k^2 - k^\sigma q_\sigma)}{(k^2 - m^2)((k+q)^2 - m^2)}.$$

Introducing the Feynman parameter x,

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{\left(xA + (1-x)B\right)^2},$$

$$\begin{aligned}\frac{1}{(k^2 - m^2)\left((k+q)^2 - m^2\right)} &= \int_0^1 \mathrm{d}x \frac{1}{\left((k+q)^2 x - m^2 x + (1-x)(k^2 - m^2)\right)^2} \\ &= \int_0^1 \mathrm{d}x \frac{1}{(k^2 + 2xkq + xq^2 - m^2)^2} \\ &= \int_0^1 \mathrm{d}x \frac{1}{(\ell^2 + x(1-x)q^2 - m^2)^2}\end{aligned}$$

where we have shifted the loop momentum to $\ell = k + xq$, so the denominator depends only on ℓ^2. We will be more succinct and say

$$\left(\ell^2 + x(1-x)q^2 - m^2\right)^2 = (\ell^2 - \Delta)^2$$

where $\Delta = -x(1-x)q^2 + m^2$ can be thought of as an effective mass term.

In terms of ℓ, the numerator becomes

$$\begin{aligned}&2k^\mu k^\nu + k^\mu q^\nu + \eta^{\mu\nu}(m^2 - k^2 - k^\sigma q_\sigma) \\ &\qquad = 2\ell^\mu \ell^\nu - \eta^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2) \\ &\qquad\quad + (\text{terms linear in } \ell).\end{aligned}$$

Now $\mathrm{d}^d k = \mathrm{d}^d \ell$ and we can rewrite our momentum integral. The Minkowski metric is preventing us from integrating over d-dimensional spherical coordinates, so we will Wick-rotate the contour by defining the Euclidean d-momentum $\ell_E = (\ell_E^0, \boldsymbol{\ell}_E)$ as

$$\ell^0 \equiv i\ell_E^0; \qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E.$$

We have the momentum line element $\ell^2 = \ell_0^2 - \boldsymbol{\ell}^2$, so $\ell^2 \to -\ell_E^2$ for $\ell_E^2 = \ell_0^2 + \boldsymbol{\ell}^2$ to have a Euclidean signature.

Unsuppressing Feynman's $i\epsilon$-prescription, we have poles at

$$(\ell^2 - \Delta + i\epsilon)^2 = 0 \qquad \Longleftrightarrow \qquad \ell_0 = \pm\left(\sqrt{\boldsymbol{\ell}^2 + \Delta} - i\epsilon\right).$$

In terms of Euclidean momentum, the denominator becomes $(-\ell_E^2 - \Delta)^2 = (\ell_E^2 + \Delta)^2$. We can again suppress the $i\epsilon$ as we are far from the poles.

Now our amplitude is also being integrated over the Feynman parameter:

$$i\Pi_2^{\mu\nu}(q) = -4ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \cdot \frac{2\ell_E^\mu \ell_E^\nu + \eta^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2}.$$

We have dropped the terms linear in ℓ_E because, for the n^{th} root of a denominator D^n only dependent on the magnitude of ℓ,

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^\mu}{D^n} = 0$$

by symmetry. A similar line of reasoning also give us

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{D^n} \neq 0 \qquad \Longleftrightarrow \qquad \mu = \nu.$$

Contracting the numerator with the metric gives $\eta_{\mu\nu}\ell^\mu\ell^\nu = \ell_\nu \ell^\nu = \ell^2$, but $\eta_{\mu\nu}\eta^{\mu\nu}\ell^2 = \eta_\mu^\mu \ell^2 = \delta_\mu^\mu \ell^2 = d\ell^2$, so we must divide by d to maintain Lorentz invariance. Then

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{D^n} = \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\frac{1}{d}\eta^{\mu\nu}\ell^2}{D^n}$$

and our amplitude looks like

$$i\Pi_2^{\mu\nu}(q) = -4ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \cdot \frac{-\frac{2}{d}\eta^{\mu\nu}\ell_E^2 + \eta^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2}.$$

In order to attack this, let's split up the momentum integral.

$$\begin{aligned}
&\int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{-\frac{2}{d}\eta^{\mu\nu}\ell_E^2 + \eta^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2} \\
&\quad = \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{(-\frac{2}{d}+1)\eta^{\mu\nu}\ell_E^2}{(\ell_E^2+\Delta)^2} \\
&\qquad + \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{-2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2}.
\end{aligned}$$

The second integral is simpler, so we'll solve it now. Consider

$$\int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^n} = \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \cdot \int_0^\infty \mathrm{d}\ell_E \, \frac{\ell_E^{d-1}}{(\ell_E^2 + \Delta)^n},$$

where Ω_d is the surface area of a d-dimensional unit sphere S^d and ℓ_E^{d-1} is the radial contribution of the Jacobian. Since the Gaussian integral gives us

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2} = \sqrt{\pi},$$

we get

$$\begin{aligned}
\pi^{\frac{d}{2}} &= \left(\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2} \right)^d \\
&= \int \mathrm{d}^d x \, \exp\left(-\sum_{i=1}^d x_i^2 \right) \\
&= \int \mathrm{d}\Omega_d \int_0^\infty \mathrm{d}x \, x^{d-1} e^{-x^2}.
\end{aligned}$$

The x integral can be solved in terms of the Γ function by substituting for x^2,

$$\begin{aligned}
\int \mathrm{d}\Omega_d \int_0^\infty \mathrm{d}x \, x^{d-1} e^{-x^2} &= \int \mathrm{d}\Omega_d \int_0^\infty \frac{\mathrm{d}u}{2x} x^{d-1} e^{-x^2} \\
&= \int \mathrm{d}\Omega_d \cdot \tfrac{1}{2} \int_0^\infty \mathrm{d}u \, x^{d-2} e^{-x^2} \\
&= \int \mathrm{d}\Omega_d \cdot \tfrac{1}{2} \int_0^\infty \mathrm{d}u \, u^{\frac{d}{2}-1} e^{-u} \\
&= \int \mathrm{d}\Omega_d \cdot \tfrac{1}{2} \Gamma(\tfrac{d}{2}),
\end{aligned}$$

so the surface area of the unit sphere in d dimensions is

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

The second factor can be solved by first making a similar substitution:

$$\int_0^\infty \mathrm{d}\ell_E \, \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^n} = \frac{1}{2}\int_0^\infty \mathrm{d}u \frac{u^{\frac{d}{2}-1}}{(u+\Delta)^n}.$$

Let $x = \frac{\Delta}{u+\Delta}$. Then $u = \Delta(\frac{1}{x} - 1)$ and

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \mathrm{d}u \, \frac{u^{\frac{d}{2}-1}}{(u+\Delta)^n} &= \frac{1}{2}\int_0^\infty \mathrm{d}u \, \frac{\Delta^{\frac{d}{2}-1}(\frac{1}{x}-1)^{\frac{d}{2}-1}}{(\Delta(\frac{1}{x}-1)+\Delta)^n} \\
&= \frac{1}{2}\Delta^{\frac{d}{2}-1}\int_0^\infty \mathrm{d}u \, \frac{(\frac{1}{x}-1)^{\frac{d}{2}-1}}{\Delta^n(\frac{1}{x})^n} \\
&= \frac{1}{2}\Delta^{\frac{d}{2}-1}\Delta^{-n}\int_0^\infty \mathrm{d}u \, x^n(\frac{1}{x}-1)^{\frac{d}{2}-1} \\
&= \frac{1}{2}\Delta^{\frac{d}{2}-n-1}\int_0^\infty \mathrm{d}u \, x^n x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1} \\
&= -\frac{1}{2}\Delta^{\frac{d}{2}-n-1}\int_1^0 \mathrm{d}x \, \Delta x^{-2} x^n x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1} \\
&= \frac{1}{2}\Delta^{\frac{d}{2}-n}\int_0^1 \mathrm{d}x \, x^{n-\frac{d}{2}-1}(1-x)^{\frac{d}{2}-1}.
\end{aligned}$$

But the beta function $B(\alpha,\beta)$ is defined in terms of the beta integral as

$$\int_0^1 \mathrm{d}x \, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

so

$$\begin{aligned}
\int_0^1 \mathrm{d}x \, x^{n-\frac{d}{2}-1}(1-x)^{\frac{d}{2}-1} &= B\left(n-\tfrac{d}{2}, \tfrac{d}{2}\right) \\
&= \frac{\Gamma(n-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(n)}.
\end{aligned}$$

Finally putting this together, we get the all important

$$\begin{aligned}
\int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} &= \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \cdot \int_0^\infty \mathrm{d}\ell_E \, \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^n} \\
&= \frac{1}{(2\pi)^d} \cdot \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \cdot \frac{1}{2}\Delta^{\frac{d}{2}-n}\frac{\Gamma(n-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(n)} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\Delta^{\frac{d}{2}-n}.
\end{aligned}$$

This means that the second term in our momentum integral (with $n = 2$) is

$$\begin{aligned}
&\int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d}\frac{-2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2+x(1-x)q^2)}{(\ell_E^2+\Delta)^2} \\
&= \Big[-2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2+x(1-x)q^2)\Big]\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\Delta^{\frac{d}{2}-2}.
\end{aligned}$$

Consequently,

$$\int \frac{\mathrm{d}\ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell_E + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \Delta^{\frac{d}{2}-n+1}$$

so the first term in our momentum integral ($n = 2$) is

$$\begin{aligned}
\int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d} \frac{(-\frac{2}{d}+1)\eta^{\mu\nu}\ell_E^2}{(\ell_E^2 + \Delta)^2} &= \left(-\frac{2}{d}+1\right)\eta^{\mu\nu} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-1} \\
&= \frac{-1+\frac{d}{2}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-1}\eta^{\mu\nu} \\
&= \frac{1-\frac{d}{2}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-2}(-\Delta\eta^{\mu\nu}) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-2}(-\Delta\eta^{\mu\nu}),
\end{aligned}$$

using the recursion relation $\Gamma(z+1) = z\Gamma(z)$.

We are now in a position to evaluate the momentum integral.

$$\begin{aligned}
&\int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d} \frac{-\frac{2}{d}\eta^{\mu\nu}\ell_E^2 + \eta^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2} \\
&= \int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d} \frac{(-\frac{2}{d}+1)\eta^{\mu\nu}\ell_E^2}{(\ell_E^2 + \Delta)^2} \\
&\quad + \int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d} \frac{-2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2 + \Delta)^2} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-2}(-\Delta\eta^{\mu\nu}) \\
&\quad + \left[-2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}\left(m^2 + x(1-x)q^2\right)\right] \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \Delta^{\frac{d}{2}-2} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \left[-\Delta\eta^{\mu\nu} - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}\left(m^2 + x(1-x)q^2\right)\right] \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \left[\eta^{\mu\nu}\left(x(1-x)q^2 - m^2\right) + \eta^{\mu\nu}\left(m^2 + x(1-x)q^2\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - 2x(1-x)q^\mu q^\nu\right] \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} 2x(1-x)\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right).
\end{aligned}$$

We are still integrating over the Feynman parameter, so the polarisation tensor is

$$\begin{aligned}
i\Pi_2^{\mu\nu} &= -4ie^2 \int_0^1 \mathrm{d}x \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} 2x(1-x) \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right) \\
&= -\frac{8ie^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x\, x(1-x) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right) \\
&= \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right) \cdot i\Pi_2(q^2)
\end{aligned}$$

where we have separated the tensor part and bundled most of the physics into

$$\Pi_2(q^2) = -\frac{8e^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x\, x(1-x) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}.$$

$\Gamma(z)$ is meromorphic in $\mathbb{C}$, with simple poles at $z \in -\mathbb{N}$. Thus, in 4-dimensional spacetime our amplitude is strongly ultraviolet-divergent. This violates the Ward identity. Pauli–Villars regularisation fails since, for a momentum cutoff $\ell_E = \Lambda$, it would give a photon mass of $m_\gamma \propto e\Lambda$ which becomes infinite once the regulator is removed.

We can instead use dimensional regularisation by calculating Π_2 in a spacetime with arbitrary dimension $d \in \mathbb{C}$, taking the limit $d \to 4$.

The Weierstrass definition of the Γ function is

$$\Gamma(z) = \frac{e^{-\gamma x}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{-\frac{z}{n}}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant.

This means we can expand Γ around $d = 4$ as

$$\Gamma(2 - \tfrac{d}{2}) = \Gamma(\tfrac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

for $\epsilon = 4 - d$.

We also get

$$\begin{aligned}
\Delta^{\frac{d}{2}-2} = \Delta^{-\frac{\epsilon}{2}} &= \exp\left\{-\frac{\epsilon}{2}\log(\Delta)\right\} \\
&= \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\log(\Delta)\right)^n}{n!} \epsilon^n \\
&= 1 - \frac{\epsilon}{2}\log(\Delta) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

and

$$(4\pi)^{-\frac{d}{2}} = (4\pi)^{\frac{\epsilon}{2}-2} = \frac{(4\pi)^{\frac{\epsilon}{2}}}{(4\pi)^2} = \frac{1}{(4\pi)^2}\exp\left\{\frac{\epsilon}{2}\log(4\pi)\right\}$$
$$= \frac{1}{(4\pi)^2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\log(4\pi)\right)^n}{n!}\epsilon^n$$
$$= \frac{1}{(4\pi)^2}\left(1 + \frac{\epsilon}{2}\log(4\pi) + \mathcal{O}(\epsilon^2)\right).$$

Then

$$\Pi_2(q^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} - \log(\Delta) - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\right).$$

This no longer violates the Ward identity, but the integral is still divergent in the $d \to 4$ limit.

However, if $d = 2$ (i.e. $1+1$ dimensional spacetime) we can recover a convergent amplitude. Note that our trace identities made use of $\mathrm{tr}(\mathbb{1}) = 4$. If $d = 2$, $\mathrm{tr}(\mathbb{1}) = 2^{\frac{d}{2}} = 2$ and

$$\Pi_2(q^2) = -\frac{4e^2}{4\pi}\int_0^1 \mathrm{d}x\, x(1-x)\frac{\Gamma(1)}{\Delta}$$
$$= -\frac{e^2}{\pi}\int_0^1 \mathrm{d}x\, \frac{x(1-x)}{-x(1-x)q^2 + m^2},$$

which is finite.

¶ Corrections to fionn@maths.tcd.ie.