Noether's Theorem and Massive Electromagnetic Fields

By Noether's theorem, there exists a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi^a - \eta^{\mu\nu}\mathcal{L}.$$

In order to ensure symmetry of $T^{\mu\nu} = T^{\nu\mu}$ (required for conservation of angular momentum), we may have to add a total divergence:

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$$

where $\mathcal{K}^{[\lambda\mu]\nu}(\phi, \partial\phi)$ is some 3-index Lorentz tensor antisymmetric in λ, μ.

Regardless of the form of $\mathcal{K}^{[\lambda\mu]\nu}$,

$$\partial_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu}_{\text{Noether}}, \qquad P^\mu_{\text{net}} = \int \mathrm{d}^3x\, T^{0\mu} = \int \mathrm{d}^3x\, T^{0\mu}_{\text{Noether}}.$$

To see this, we take the 4-divergence

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu \left[T^{\mu\nu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \right] \\ &= \partial_\mu T^{\mu\nu}_{\text{Noether}} + \partial_\mu \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}. \end{aligned}$$

Since $\partial_\mu \partial_\lambda$ is symmetric and $\mathcal{K}^{[\lambda\mu]\nu}$ antisymmetric in λ, μ, we conclude that $\partial_\mu \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} = 0$. To check conservation,

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu \left[\sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi^a - \eta^{\mu\nu}\mathcal{L} \right] \\ &= \sum_a \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi^a \right] - \partial^\nu \mathcal{L} \\ &= \sum_a \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \partial^\nu \phi^a \right] - \partial^\nu \mathcal{L} \\ &= \sum_a \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \partial^\nu \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \partial^\nu \phi^a \right] - \partial^\nu \mathcal{L}, \end{aligned}$$

but

$$\partial^\nu \mathcal{L} = \sum_a \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \partial^\nu \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \partial^\nu \phi^a \right],$$

so the derivative becomes

$$\partial_\mu T^{\mu\nu} = \partial^\nu \mathcal{L} - \partial^\nu \mathcal{L} = 0$$

and the tensor is conserved.

Now

$$\begin{aligned} P^\mu_{\text{net}} &= \int \mathrm{d}^3x \, T^{0\mu} \\ &= \int \mathrm{d}^3x \, \left[T^{0\mu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda 0]\mu} \right], \end{aligned}$$

so we wish to show that $\int \mathrm{d}^3x \, \partial_\lambda \mathcal{K}^{[\lambda 0]\mu} = 0$.

$$\begin{aligned} \int \mathrm{d}^3x \, \partial_\lambda \mathcal{K}^{[\lambda 0]\mu} &= \int \mathrm{d}^3x \, \left(\partial_0 \mathcal{K}^{[00]\mu} + \partial_i \mathcal{K}^{[i0]\mu} \right) \\ &= \int \mathrm{d}^3x \, \partial_i \mathcal{K}^{[i0]\mu} \end{aligned}$$

by symmetry and

$$\int \mathrm{d}^3x \, \partial_i \mathcal{K}^{[i0]\mu} = 0$$

when integrated over a surface.

$T^{\mu\nu}_{\text{Noether}}$ is symmetric for real or complex scalar fields but loses this property for vector, tensor or spinor fields. This is the case for a theory of free electromagnetic fields, where

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

With this Lagrangian,

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \eta^{\mu\nu} \mathcal{L} \\ &= -\frac{1}{4} \frac{\partial(F_{\rho\sigma} F^{\rho\sigma})}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \end{aligned}$$

But

$$F_{\rho\sigma} F^{\rho\sigma} = (\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)$$

$$
\begin{aligned}
&= \partial_\rho A_\sigma \partial^\rho A^\sigma - \partial_\rho A_\sigma \partial^\sigma A^\rho - \partial_\sigma A_\rho \partial^\rho A^\sigma + \partial_\sigma A_\rho \partial^\sigma A^\rho \\
&= 2\partial_\rho A_\sigma \partial^\rho A^\sigma - 2\partial_\rho A_\sigma \partial^\sigma A^\rho \\
&= 2\partial_\rho A_\sigma (\partial^\rho A^\sigma - \partial^\sigma A^\rho),
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\lambda)} &= 2\frac{\partial(\partial_\rho A_\sigma)}{\partial(\partial_\mu A_\lambda)}(\partial^\rho A^\sigma - \partial^\sigma A^\rho) + 2\partial_\rho A_\sigma \frac{\partial(\partial^\rho A^\sigma - \partial^\sigma A^\rho)}{\partial(\partial_\mu A_\lambda)} \\
&= 2\delta^\mu_\rho \delta^\lambda_\sigma (\partial^\rho A^\sigma - \partial^\sigma A^\rho) + 2\partial^\rho A^\sigma \frac{\partial(\partial_\rho A_\sigma - \partial_\sigma A_\rho)}{\partial(\partial_\mu A_\lambda)} \\
&= 2\delta^\mu_\rho \delta^\lambda_\sigma (\partial^\rho A^\sigma - \partial^\sigma A^\rho) + 2\partial^\rho A^\sigma (\delta^\mu_\rho \delta^\lambda_\sigma - \delta^\mu_\sigma \delta^\lambda_\rho) \\
&= 2(\partial^\mu A^\lambda - \partial^\lambda A^\mu) + 2(\partial^\mu A^\lambda - \partial^\lambda A^\mu) \\
&= 4F^{\mu\lambda},
\end{aligned}
$$

giving

$$
T^{\mu\nu}_{\text{Noether}} = -F^{\mu\lambda}\partial^\nu A_\lambda + \tfrac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.
$$

This stress tensor is neither symmetric nor gauge invariant. The second term is obviously symmetric and gauge invariant, since $\eta^{\mu\nu}$ and $F_{\mu\nu}F^{\mu\nu}$ are both symmetric and gauge invariant, so we will focus on the first term.

$$
\begin{aligned}
F^{\mu\lambda}\partial^\nu A_\lambda &\xrightarrow{\lambda\leftrightarrow\nu} F^{\mu\nu}\partial^\lambda A_\nu \\
(\partial^\mu A^\lambda - \partial^\lambda A^\mu)\partial^\nu A_\lambda &\xrightarrow{\lambda\leftrightarrow\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu)\partial^\lambda A_\nu \\
\partial^\mu A^\lambda \partial^\nu A_\lambda - \cancel{\partial^\lambda A^\mu \partial^\nu A_\lambda} &\xrightarrow{\lambda\leftrightarrow\nu} \partial^\mu A^\nu \partial^\lambda A_\lambda - \cancel{\partial^\nu A^\mu \partial^\lambda A_\nu}
\end{aligned}
$$

and $\partial^\mu A^\lambda \partial^\nu A_\lambda \neq \partial^\mu A^\nu \partial^\lambda A_\lambda$, so $T^{\mu\nu}_{\text{Noether}}$ is asymmetric. Under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$,

$$
\begin{aligned}
(\partial_\mu A_\lambda - \partial_\lambda A_\mu)\partial^\nu A_\lambda &\to (\partial_\mu A_\lambda - \partial_\lambda A_\mu + \partial_\mu\partial_\lambda\alpha - \partial_\lambda\partial_\mu\alpha)(\partial^\nu A_\lambda + \partial^\nu\partial_\lambda\alpha) \\
&= (\partial_\mu A_\lambda - \partial_\lambda A_\mu)(\partial^\nu A_\lambda + \partial^\nu\partial_\lambda\alpha)
\end{aligned}
$$

and

$$
(\partial_\mu A_\lambda - \partial_\lambda A_\mu)\partial^\nu A_\lambda \neq (\partial_\mu A_\lambda - \partial_\lambda A_\mu)(\partial^\nu A_\lambda + \partial^\nu\partial_\lambda\alpha),
$$

so $T^{\mu\nu}_{\text{Noether}}$ is not gauge invariant.

A candidate symmetric and gauge invariant stress-energy tensor is

$$
\Theta^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda + \tfrac{1}{4}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}.
$$

However, it remains to be shown that this has the form of

$$
\Theta^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}.
$$

Assume this to be the case,

$$-F^{\mu\lambda}F^{\nu}{}_{\lambda}+\tfrac{1}{4}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}=-F^{\mu\lambda}\partial^{\nu}A_{\lambda}+\tfrac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}+\partial_{\lambda}\mathcal{K}^{[\lambda\mu]\nu}$$
$$-F^{\mu\lambda}F^{\nu}{}_{\lambda}=-F^{\mu\lambda}\partial^{\nu}A_{\lambda}+\partial_{\lambda}\mathcal{K}^{[\lambda\mu]\nu}$$

so

$$\begin{aligned}\partial_{\lambda}\mathcal{K}^{[\lambda\mu]\nu}&=F^{\mu\lambda}\partial^{\nu}A_{\lambda}-F^{\mu\lambda}F^{\nu}{}_{\lambda}\\&=(\partial^{\mu}A^{\lambda}-\partial^{\lambda}A^{\mu})\partial^{\nu}A_{\lambda}-(\partial^{\mu}A^{\lambda}-\partial^{\lambda}A^{\mu})(\partial^{\nu}A_{\lambda}-\partial_{\lambda}A^{\nu})\\&=\partial^{\mu}A^{\lambda}\partial^{\nu}A_{\lambda}-\partial^{\lambda}A^{\mu}\partial^{\nu}A_{\lambda}-(\partial^{\mu}A^{\lambda}-\partial^{\lambda}A^{\mu})(\partial^{\nu}A_{\lambda}-\partial_{\lambda}A^{\nu})\\&=\partial^{\mu}A^{\lambda}\partial_{\lambda}A^{\nu}-\partial^{\lambda}A^{\mu}\partial_{\lambda}A^{\nu},\end{aligned}$$

showing that $\Theta^{\mu\nu}$ indeed has the desired form, for $\partial_{\lambda}\mathcal{K}^{[\lambda\mu]\nu}$ as above.

We'll now consider a massive relativistic vector field A^{μ} with Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_{\mu}A^{\mu}-A_{\mu}J^{\mu}.$$

This is the Proca Lagrangian with sources. The equations of motion are given by the Euler–Lagrange equation:

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})}-\frac{\partial\mathcal{L}}{\partial A_{\nu}}=0.$$

Taking derivatives,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial A_{\nu}}&=\frac{1}{2}m^2\eta^{\rho\lambda}\frac{\partial(A_{\rho}A_{\lambda})}{\partial A_{\nu}}-\frac{\partial A_{\rho}}{\partial A_{\nu}}J^{\rho}\\&=\frac{1}{2}m^2\eta^{\rho\lambda}(\delta^{\nu}_{\rho}A_{\lambda}+A_{\rho}\delta^{\nu}_{\lambda})-\delta^{\nu}_{\rho}J^{\rho}\\&=m^2A^{\nu}-J^{\nu}\end{aligned}$$

since $F_{\rho\sigma}F^{\rho\sigma}$ does not depend explicitly on A_{ν} and

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})}=-\partial_{\mu}F^{\mu\nu}$$

as before, since the mass and source terms are independent of $\partial_{\mu}A_{\nu}$. Thus, our massive spin-1 field obeys

$$\partial_{\mu}F^{\mu\nu}+m^2A^{\nu}-J^{\nu}=0.$$

This does not force conservation of current, as

$$\partial_{\nu}\left[\partial_{\mu}F^{\mu\nu}+m^2A^{\nu}-J^{\nu}\right]=0,$$

$$\partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu - \partial_\nu J^\nu = 0,$$
$$\partial_\nu J^\nu = \partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu \neq 0$$

necessarily.

$$\begin{aligned}\partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu &= \partial_\nu\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2\partial_\nu A^\nu \\ &= \partial_\nu\partial_\mu\partial^\mu A^\nu - \partial_\nu\partial_\mu\partial^\nu A^\mu + m^2\partial_\nu A^\nu \\ &= \partial_\mu\partial^\mu\partial_\nu A^\nu - \partial_\nu\partial^\nu\partial_\mu A^\mu + m^2\partial_\nu A^\nu \\ &= (\partial_\mu\partial^\mu - \partial_\mu\partial^\mu + m^2)\partial_\nu A^\nu \\ &= m^2\partial_\nu A^\nu\end{aligned}$$

so if $\partial_\nu J^\nu = 0$, $\partial_\mu A^\mu = 0$ since $m^2 \neq 0$ necessarily. This is the Lorenz gauge condition, which we expected to emerge since the Proca Lagrangian is the Stueckelberg Lagrangian with gauge fixing. Going backwards a little bit, we can remove a term using our inherited gauge condition to get

$$\begin{aligned}\partial_\nu\partial_\mu\partial^\mu A^\nu + m^2\partial_\nu A^\nu &= \partial_\nu J^\nu \\ &= \partial_\nu\left[\left(\partial_\mu\partial^\mu + m^2\right)A^\nu\right] \\ &= \partial_\nu\left(\partial_\mu\partial^\mu + m^2\right)A^\nu,\end{aligned}$$

so on integrating,

$$\left(\Box + m^2\right)A^\nu = J^\nu,$$

the Klein–Gordon equation with sources.

☼ Corrections to `fionnf@maths.tcd.ie`.