Massive Gauge Field

The Lagrangian density

$$\begin{aligned}\mathcal{L}(\phi^\mu) &= -\frac{1}{2}(\partial_\mu\phi^\nu)^2 + \frac{1}{2}(\partial_\mu\phi^\mu)^2 + \frac{m^2}{2}\phi_\mu\phi^\mu \\ &= -\frac{1}{2}\partial_\mu\phi^\nu\partial^\mu\phi_\nu + \frac{1}{2}\partial_\mu\phi^\mu\partial_\nu\phi^\nu + \frac{m^2}{2}\phi_\mu\phi^\mu\end{aligned}$$

describes a massive gauge field.

The field equations are derived via Euler–Lagrange*:

$$\partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} - \frac{\partial\mathcal{L}}{\partial\phi^\rho} = 0. \tag{$\mathcal{E}$–$\mathcal{L}$}$$

I'm going to go slowly here; we'll pick up speed later.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi^\rho} &= \frac{\partial}{\partial\phi^\rho}\left[\frac{m^2}{2}\phi_\mu\phi^\mu\right] \\ &= \frac{m^2}{2}\left[\left(\frac{\partial}{\partial\phi^\rho}\phi_\mu\right)\phi^\mu + \phi_\mu\left(\frac{\partial}{\partial\phi^\rho}\phi^\mu\right)\right] \\ &= \frac{m^2}{2}\left[\left(\frac{\partial}{\partial\phi^\rho}\eta_{\mu\nu}\phi^\nu\right)\phi^\mu + \phi_\mu\delta^\mu_\rho\right] \\ &= \frac{m^2}{2}\left[\eta_{\mu\nu}\left(\frac{\partial}{\partial\phi^\rho}\phi^\nu\right)\phi^\mu + \phi_\mu\delta^\mu_\rho\right] \\ &= \frac{m^2}{2}\left[\eta_{\mu\nu}\delta^\nu_\rho\phi^\mu + \phi_\mu\delta^\mu_\rho\right] \\ &= \frac{m^2}{2}\left[\eta_{\mu\rho}\phi^\mu + \phi_\rho\right] \\ &= \frac{m^2}{2}\left[\phi_\rho + \phi_\rho\right] = m^2\phi_\rho.\end{aligned}$$

*It seems natural to choose $\partial\phi^\rho$ rather than $\partial\phi_\rho$, but I appear to be going against the "literature" here. I remember differentiating against A_μ rather than A^μ in the $\mathcal{E}$–$\mathcal{L}$ equations in classical field theory, but it doesn't really matter. Also, *: looks like a face.

That was fun! We could have arrived at this much faster by just saying $\phi_\mu\phi^\mu = \phi^2$, so $\frac{\partial}{\partial\phi^\rho}\phi^2 = 2\phi_\rho$, where the lower ρ index is determined by the upper ρ on $\partial\phi^\rho$.

The next term is a bit trickier.

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} &= \frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[-\frac{1}{2}(\partial_\mu\phi^\nu)^2 + \frac{1}{2}(\partial_\mu\phi^\mu)^2\right] \\
&= \frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[-\frac{1}{2}\partial_\mu\phi^\nu\partial^\mu\phi_\nu + \frac{1}{2}\partial_\mu\phi^\mu\partial_\nu\phi^\nu\right] \\
&= -\frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[\partial_\mu\phi^\nu\partial^\mu\phi_\nu\right] + \frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[\partial_\mu\phi^\mu\partial_\nu\phi^\nu\right] \\
&= -\frac{1}{2}\left[\frac{\partial(\partial_\mu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)}\partial^\mu\phi_\nu + \partial_\mu\phi^\nu\frac{\partial(\partial^\mu\phi_\nu)}{\partial(\partial_\lambda\phi^\rho)}\right] \\
&\quad + \frac{1}{2}\left[\frac{\partial(\partial_\mu\phi^\mu)}{\partial(\partial_\lambda\phi^\rho)}\partial_\nu\phi^\nu + \partial_\mu\phi^\mu\frac{\partial(\partial_\nu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)}\right] \\
&= -\frac{1}{2}\left[\frac{\partial(\partial_\mu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)}\partial^\mu\phi_\nu + \partial_\mu\phi^\nu\eta^{\mu\alpha}\eta_{\nu\beta}\frac{\partial(\partial_\alpha\phi^\beta)}{\partial(\partial_\lambda\phi^\rho)}\right] \\
&\quad + \frac{1}{2}\left[\frac{\partial(\partial_\mu\phi^\mu)}{\partial(\partial_\lambda\phi^\rho)}\partial_\nu\phi^\nu + \partial_\mu\phi^\mu\frac{\partial(\partial_\nu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)}\right] \\
&= -\frac{1}{2}\left[\delta^\lambda_\mu\delta^\nu_\rho\partial^\mu\phi_\nu + \partial_\mu\phi^\nu\eta^{\mu\alpha}\eta_{\nu\beta}\delta^\lambda_\alpha\delta^\beta_\rho\right] \\
&\quad + \frac{1}{2}\left[\delta^\lambda_\mu\delta^\mu_\rho\partial_\nu\phi^\nu + \partial_\mu\phi^\mu\delta^\lambda_\nu\delta^\nu_\rho\right] \\
&= -\frac{1}{2}\left[\delta^\lambda_\mu\delta^\nu_\rho\partial^\mu\phi_\nu + \partial_\mu\phi^\nu\eta^{\mu\lambda}\eta_{\nu\rho}\right] \\
&\quad + \frac{1}{2}\left[\delta^\lambda_\mu\delta^\mu_\rho\partial_\nu\phi^\nu + \partial_\mu\phi^\mu\delta^\lambda_\nu\delta^\nu_\rho\right] \\
&= -\frac{1}{2}\left[\partial^\lambda\phi_\rho + \partial^\lambda\phi_\rho\right] + \frac{1}{2}\left[\delta^\lambda_\rho\partial_\nu\phi^\nu + \partial_\mu\phi^\mu\delta^\lambda_\rho\right] \\
&= -\partial^\lambda\phi_\rho + \delta^\lambda_\rho\partial_\mu\phi^\mu.
\end{aligned}
$$

If we used a lower ρ in our definition (see footnote $*$), we'd have a metric tensor instead of a Kronecker delta. The fast way:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} &= \frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[-\frac{1}{2}(\partial_\mu\phi^\nu)^2 + \frac{1}{2}(\partial_\mu\phi^\mu)^2\right] \\
&= -\frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}(\partial_\mu\phi^\nu)^2 + \frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}(\partial_\mu\phi^\mu)^2 \\
&= -\partial_\mu\phi^\nu\frac{\partial(\partial_\mu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)} + \partial_\mu\phi^\mu\frac{\partial(\partial_\nu\phi^\nu)}{\partial(\partial_\lambda\phi^\rho)}
\end{aligned}
$$

$$
\begin{aligned}
&= -\partial_\mu \phi^\nu \delta^\lambda_\mu \delta^\nu_\rho + \partial_\mu \phi^\mu \delta^\lambda_\nu \delta^\nu_\rho \\
&= -\eta^{\lambda\kappa}\eta_{\rho\sigma}\partial_\mu \phi^\nu \delta_{\mu\kappa}\delta^{\nu\sigma} + \partial_\mu \phi^\mu \delta^\lambda_\rho \\
&= -\eta^{\lambda\kappa}\eta_{\rho\sigma}\partial_\kappa \phi^\sigma + \delta^\lambda_\rho \partial_\mu \phi^\mu \\
&= -\partial^\lambda \phi_\rho + \delta^\lambda_\rho \partial_\mu \phi^\mu .
\end{aligned}
$$

We are compelled to take the derivative ∂_λ of this.

$$
\begin{aligned}
\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi^\rho)} &= \partial_\lambda \left[-\partial^\lambda \phi_\rho + \delta^\lambda_\rho \partial_\mu \phi^\mu \right] \\
&= -\partial_\lambda \partial^\lambda \phi_\rho + \partial_\lambda \delta^\lambda_\rho \partial_\mu \phi^\mu \\
&= -\Box \phi_\rho + \partial_\rho \partial_\mu \phi^\mu .
\end{aligned}
$$

Therefore, by ($\mathcal{E}$–$\mathcal{L}$) the field equations are

$$
-\Box \phi_\rho + \partial_\rho \partial_\mu \phi^\mu - m^2 \phi_\rho = 0.
$$

The derivative of the equations of motion gives

$$
\begin{aligned}
\partial_\rho \left[-\Box \phi_\rho + \partial_\rho \partial_\mu \phi^\mu - m^2 \phi_\rho \right] &= 0 \\
m^2 \partial_\mu \phi^\mu &= 0 \\
\partial_\mu \phi^\mu &= 0 \qquad \because m^2 \neq 0,
\end{aligned}
$$

so the field is conserved.

Now we'll construct the canonical momenta conjugate to ϕ_μ.

$$
\begin{aligned}
\pi_\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\mu)} \\
&= -\partial^0 \phi_\mu + \delta^0_\mu \partial^\sigma \phi_\sigma .
\end{aligned}
$$

This allows us to determine the Hamiltonian.

$$
\begin{aligned}
\mathcal{H} &= \pi_\mu \partial_0 \phi^\mu - \mathcal{L} \\
&= -\partial_0 \phi^\mu \partial^0 \phi_\mu + \delta^0_\mu \partial_0 \phi^\mu \partial^\sigma \phi_\sigma - \mathcal{L} \\
&= -(\partial_0 \phi^\mu)^2 + \partial_0 \phi^0 \partial^\sigma \phi_\sigma + \frac{1}{2}(\partial_\mu \phi^\nu)^2 - \frac{1}{2}(\partial_\mu \phi^\mu)^2 - \frac{m^2}{2}\phi_\mu \phi^\mu ,
\end{aligned}
$$

$$
H = \int \mathrm{d}^3 x \left[\frac{1}{2}(\partial_\mu \phi^\nu)^2 - \frac{1}{2}(\partial_\mu \phi^\mu)^2 - \frac{m^2}{2}\phi_\mu \phi^\mu - (\partial_0 \phi^\mu)^2 + \partial_0 \phi^0 \partial^\sigma \phi_\sigma \right].
$$

This theory is not invariant under the gauge transformation $\phi^\mu \to \phi^\mu + \partial^\mu \alpha$. To see why, we must study the dynamics of the transformed field. We have that

$$
\mathcal{L}(\phi) = -\frac{1}{2}\partial_\mu \phi^\nu \partial^\mu \phi_\nu + \frac{1}{2}\partial_\mu \phi^\mu \partial_\nu \phi^\nu + \frac{m^2}{2}\phi_\mu \phi^\mu ,
$$

so under a gauge transformation $\phi \to \phi'$ our Lagrangian $\mathcal{L}(\phi')$ will read

$$\begin{aligned}
\mathcal{L}(\phi') =& -\frac{1}{2}\partial_\mu \left(\phi^\nu + \partial^\nu \alpha\right) \partial^\mu \left(\phi_\nu + \partial_\nu \alpha\right) + \frac{1}{2}\partial_\mu \left(\phi^\mu + \partial^\mu \alpha\right) \partial_\nu \left(\phi^\nu + \partial^\nu \alpha\right) \\
&+ \frac{m^2}{2}\left[\phi_\mu \phi^\mu + 2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right] \\
=& -\frac{1}{2}\left(\partial_\mu \phi^\nu + \partial_\mu \partial^\nu \alpha\right)\left(\partial^\mu \phi_\nu + \partial^\mu \partial_\nu \alpha\right) + \frac{1}{2}\left(\partial_\mu \phi^\mu + \partial_\mu \partial^\mu \alpha\right)\left(\partial_\nu \phi^\nu + \partial_\nu \partial^\nu \alpha\right) \\
&+ \frac{m^2}{2}\left[\phi_\mu \phi^\mu + 2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right] \\
=& -\frac{1}{2}\left[\partial_\mu \phi^\nu \partial^\mu \phi_\nu + \partial_\mu \phi^\nu \partial^\mu \partial_\nu \alpha + \partial^\mu \phi_\nu \partial_\mu \partial^\nu \alpha + \partial_\mu \partial^\nu \alpha \partial^\mu \partial_\nu \alpha\right] \\
&+ \frac{1}{2}\left[\partial_\mu \phi^\mu \partial_\nu \phi^\nu + \partial_\mu \phi^\mu \partial_\nu \partial^\nu \alpha + \partial_\nu \phi^\nu \partial_\mu \partial^\mu \alpha + \partial_\mu \partial^\mu \alpha \partial_\nu \partial^\nu \alpha\right] \\
&+ \frac{m^2}{2}\left[\phi_\mu \phi^\mu + 2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right] \\
=& -\frac{1}{2}\partial_\mu \phi^\nu \partial^\mu \phi_\nu - \frac{1}{2}\left[\partial_\mu \phi^\nu \partial^\mu \partial_\nu \alpha + \partial^\mu \phi_\nu \partial_\mu \partial^\nu \alpha + \partial_\mu \partial^\nu \alpha \partial^\mu \partial_\nu \alpha\right] \\
&+ \frac{1}{2}\partial_\mu \phi^\mu \partial_\nu \phi^\nu + \frac{1}{2}\left[\partial_\mu \phi^\mu \partial_\nu \partial^\nu \alpha + \partial_\nu \phi^\nu \partial_\mu \partial^\mu \alpha + \partial_\mu \partial^\mu \alpha \partial_\nu \partial^\nu \alpha\right] \\
&+ \frac{m^2}{2}\phi_\mu \phi^\mu + \frac{m^2}{2}\left[2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right] \\
=& \mathcal{L}(\phi) - \frac{1}{2}\left[\partial_\mu \phi^\nu \partial^\mu \partial_\nu \alpha + \partial^\mu \phi_\nu \partial_\mu \partial^\nu \alpha + \partial_\mu \partial^\nu \alpha \partial^\mu \partial_\nu \alpha\right] \\
&+ \frac{1}{2}\left[\partial_\mu \phi^\mu \partial_\nu \partial^\nu \alpha + \partial_\nu \phi^\nu \partial_\mu \partial^\mu \alpha + \partial_\mu \partial^\mu \alpha \partial_\nu \partial^\nu \alpha\right] \\
&+ \frac{m^2}{2}\left[2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right] \\
=& \mathcal{L}(\phi) + \mathcal{L}_g(\phi)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_g(\phi) =& -\frac{1}{2}\left[\partial_\mu \phi^\nu \partial^\mu \partial_\nu \alpha + \partial^\mu \phi_\nu \partial_\mu \partial^\nu \alpha + \partial_\mu \partial^\nu \alpha \partial^\mu \partial_\nu \alpha\right] \\
&+ \frac{1}{2}\left[\partial_\mu \phi^\mu \partial_\nu \partial^\nu \alpha + \partial_\nu \phi^\nu \partial_\mu \partial^\mu \alpha + \partial_\mu \partial^\mu \alpha \partial_\nu \partial^\nu \alpha\right] \\
&+ \frac{m^2}{2}\left[2\phi^\mu \partial_\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha\right].
\end{aligned}$$

The dynamics of the field ϕ' are thus determined by

$$\partial_\lambda \frac{\partial\left(\mathcal{L} + \mathcal{L}_g\right)}{\partial(\partial_\lambda \phi^\rho)} - \frac{\partial\left(\mathcal{L} + \mathcal{L}_g\right)}{\partial \phi^\rho} = 0.$$

However,

$$\begin{aligned}\partial_\lambda \frac{\partial\left(\mathcal{L}+\mathcal{L}_g\right)}{\partial(\partial_\lambda \phi^\rho)} - \frac{\partial\left(\mathcal{L}+\mathcal{L}_g\right)}{\partial\phi^\rho} &= \partial_\lambda\left[\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} + \frac{\partial\mathcal{L}_g}{\partial(\partial_\lambda\phi^\rho)}\right] - \frac{\partial\mathcal{L}}{\partial\phi^\rho} - \frac{\partial\mathcal{L}_g}{\partial\phi^\rho}\\ &= \partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} - \frac{\partial\mathcal{L}}{\partial\phi^\rho} + \partial_\lambda\frac{\partial\mathcal{L}_g}{\partial(\partial_\lambda\phi^\rho)} - \frac{\partial\mathcal{L}_g}{\partial\phi^\rho},\end{aligned}$$

so our Euler–Lagrange equation is

$$\partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^\rho)} - \frac{\partial\mathcal{L}}{\partial\phi^\rho} + \partial_\lambda\frac{\partial\mathcal{L}_g}{\partial(\partial_\lambda\phi^\rho)} - \frac{\partial\mathcal{L}_g}{\partial\phi^\rho} = 0.$$

We will examine the $\mathcal{L}_g$ terms.

$$\begin{aligned}\frac{\partial\mathcal{L}_g}{\partial\phi^\rho} &= \frac{m^2}{2}\frac{\partial}{\partial\phi^\rho}\left[2\phi^\mu\partial_\mu\alpha\right]\\ &= m^2\partial_\mu\alpha\frac{\partial\phi^\mu}{\partial\phi^\rho}\\ &= m^2\partial_\mu\alpha\delta^\mu_\rho\\ &= m^2\partial_\rho\alpha.\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{L}_g}{\partial(\partial_\lambda\phi^\rho)} =& -\frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[\partial_\mu\phi^\nu\partial^\mu\partial_\nu\alpha + \partial^\mu\phi_\nu\partial_\mu\partial^\nu\alpha + \partial_\mu\partial^\nu\alpha\partial^\mu\partial_\nu\alpha\right]\\ &+ \frac{1}{2}\frac{\partial}{\partial(\partial_\lambda\phi^\rho)}\left[\partial_\mu\phi^\mu\partial_\nu\partial^\nu\alpha + \partial_\nu\phi^\nu\partial_\mu\partial^\mu\alpha + \partial_\mu\partial^\mu\alpha\partial_\nu\partial^\nu\alpha\right]\\ =& -\frac{1}{2}\left[\partial^\mu\partial_\nu\alpha\frac{\partial\left(\partial_\mu\phi^\nu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)} + \partial_\mu\partial^\nu\alpha\frac{\partial\left(\partial^\mu\phi_\nu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)}\right]\\ &+ \frac{1}{2}\left[\partial_\nu\partial^\nu\alpha\frac{\partial\left(\partial_\mu\phi^\mu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)} + \partial_\mu\partial^\mu\alpha\frac{\partial\left(\partial_\nu\phi^\nu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)}\right]\\ =& -\frac{1}{2}\left[\partial^\mu\partial_\nu\alpha\frac{\partial\left(\partial_\mu\phi^\nu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)} + \partial_\mu\partial^\nu\alpha\eta^{\mu\alpha}\eta_{\nu\beta}\frac{\partial(\partial_\alpha\phi^\beta)}{\partial\left(\partial_\lambda\phi^\rho\right)}\right]\\ &+ \frac{1}{2}\left[\partial_\nu\partial^\nu\alpha\frac{\partial\left(\partial_\mu\phi^\mu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)} + \partial_\mu\partial^\mu\alpha\frac{\partial\left(\partial_\nu\phi^\nu\right)}{\partial\left(\partial_\lambda\phi^\rho\right)}\right]\\ =& -\frac{1}{2}\left[\partial^\mu\partial_\nu\alpha\delta^\lambda_\mu\delta^\nu_\rho + \partial_\mu\partial^\nu\alpha\eta^{\mu\alpha}\eta_{\nu\beta}\delta^\lambda_\alpha\delta^\beta_\rho\right]\\ &+ \frac{1}{2}\left[\partial_\nu\partial^\nu\alpha\delta^\lambda_\mu\delta^\mu_\rho + \partial_\mu\partial^\mu\alpha\delta^\lambda_\nu\delta^\nu_\rho\right]\\ =& -\frac{1}{2}\left[\partial^\lambda\partial_\rho\alpha + \partial^\lambda\partial_\rho\alpha\right] + \frac{1}{2}\left[\partial_\nu\partial^\nu\alpha\delta^\lambda_\rho + \partial_\mu\partial^\mu\alpha\delta^\lambda_\rho\right]\\ =& -\partial^\lambda\partial_\rho\alpha + \delta^\lambda_\rho\Box\alpha.\end{aligned}$$

This has vanishing four-divergence, as

$$\begin{aligned}\partial_\lambda \frac{\partial \mathcal{L}_g}{\partial(\partial_\lambda \phi^\rho)} &= \partial_\lambda \left[-\partial^\lambda \partial_\rho \alpha + \delta^\lambda_\rho \Box \alpha\right] \\ &= -\partial_\lambda \partial^\lambda \partial_\rho \alpha + \delta^\lambda_\rho \partial_\lambda \Box \alpha \\ &= -\Box \partial_\rho \alpha + \partial_\rho \Box \alpha = 0.\end{aligned}$$

We can now construct the equations of motion for our field. By substitution,

$$-\Box \phi_\rho + \partial_\rho \partial_\mu \phi^\mu - m^2 \phi_\rho - m^2 \partial_\rho \alpha = 0.$$

This will only collide with our original field equations if $\partial_\rho \alpha = 0$, which would not be much of a gauge transformation at all. Hence, our theory is not gauge invariant.

Corrections to fionnf@maths.tcd.ie.