SO(n) Invariant Scalar Field Theory

The Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^a \partial^\mu \Phi^a - \frac{1}{2}m^2\Phi^a\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a\right)^2$$

describes a SO(n) symmetric theory of n real scalar fields.

The classical field equations are derived via

$$\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Phi^b)} - \frac{\partial \mathcal{L}}{\partial \Phi^b} = 0. \qquad (\mathcal{E}\text{–}\mathcal{L})$$

Taking it slowly,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \Phi^b} &= \frac{\partial}{\partial \Phi^b}\left[-\frac{1}{2}m^2\Phi^a\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a\right)^2\right] \\
&= -\frac{1}{2}m^2\frac{\partial}{\partial\Phi^b}\left(\Phi^a\Phi^a\right) - \frac{1}{4}\lambda\frac{\partial}{\partial\Phi^b}\left(\Phi^a\Phi^a\right)^2 \\
&= -\frac{1}{2}m^2\left(\Phi^a\frac{\partial\Phi^a}{\partial\Phi^b} + \frac{\partial\Phi^a}{\partial\Phi^b}\Phi^a\right) - \frac{1}{2}\lambda\Phi^c\Phi^c\frac{\partial}{\partial\Phi^b}\left(\Phi^a\Phi^a\right) \\
&= -\frac{1}{2}m^2\left(\Phi^a\frac{\partial\Phi^a}{\partial\Phi^b} + \frac{\partial\Phi^a}{\partial\Phi^b}\Phi^a\right) - \frac{1}{2}\lambda\Phi^c\Phi^c\left(\Phi^a\frac{\partial\Phi^a}{\partial\Phi^b} + \frac{\partial\Phi^a}{\partial\Phi^b}\Phi^a\right) \\
&= -\frac{1}{2}m^2\left(\Phi^a\delta^{ab} + \delta^{ab}\Phi^a\right) - \frac{1}{2}\lambda\Phi^c\Phi^c\left(\Phi^a\delta^{ab} + \delta^{ab}\Phi^a\right) \\
&= -m^2\Phi^b - \lambda\Phi^c\Phi^c\Phi^b = -m^2\Phi^b - \lambda\Phi^a\Phi^a\Phi^b.
\end{aligned}$$

Up next is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\lambda\Phi^b)} &= \frac{\partial}{\partial(\partial_\lambda\Phi^b)}\left[\frac{1}{2}\partial_\mu\Phi^a\partial^\mu\Phi^a\right] \\
&= \frac{1}{2}\left[\partial_\mu\Phi^a\frac{\partial(\partial^\mu\Phi^a)}{\partial(\partial_\lambda\Phi^b)} + \frac{\partial(\partial_\mu\Phi^a)}{\partial(\partial_\lambda\Phi^b)}\partial^\mu\Phi^a\right] \\
&= \frac{1}{2}\left[\partial_\mu\Phi^a\eta^{\mu\kappa}\frac{\partial(\partial_\kappa\Phi^a)}{\partial(\partial_\lambda\Phi^b)} + \frac{\partial(\partial_\mu\Phi^a)}{\partial(\partial_\lambda\Phi^b)}\partial^\mu\Phi^a\right] \\
&= \frac{1}{2}\left[\partial_\mu\Phi^a\eta^{\mu\kappa}\delta^\lambda_\kappa\delta^{ab} + \delta^\lambda_\mu\delta^{ab}\partial^\mu\Phi^a\right] \\
&= \frac{1}{2}\left[\partial_\mu\Phi^b\eta^{\mu\lambda} + \partial^\lambda\Phi^b\right] = \partial^\lambda\Phi^b.
\end{aligned}$$

Taking the 4-divergence ∂_λ,

$$\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Phi^b)} = \partial_\lambda \partial^\lambda \Phi^b = \Box \Phi^b.$$

Thus, the field equations from ($\mathcal{E}$–$\mathcal{L}$) are

$$\Box \Phi^b + m^2 \Phi^b + \lambda \Phi^a \Phi^a \Phi^b = 0$$

or

$$\left(\Box + m^2 + \lambda \Phi^b \Phi^b\right) \Phi^a = 0.$$

In order to study the Hamiltonian $H = \int \mathrm{d}^3 x\, \mathcal{H}$, we must introduce the canonical momenta π^a satisfying

$$\left\{\pi^a(x), \Phi^b(y)\right\}_{\mathrm{P}} = \delta^{ab} \delta^{(3)}(x - y).$$

π^a is defined as

$$\pi^a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi^a)} = \partial^0 \Phi^a,$$

so

$$\begin{aligned} \mathcal{H} &= \pi^a \dot{\Phi}^a - \mathcal{L} = \partial^0 \Phi^a \partial_0 \Phi^a - \mathcal{L} \\ &= \partial^0 \Phi^a \partial_0 \Phi^a - \frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a + \frac{1}{2} m^2 \Phi^a \Phi^a + \frac{1}{4} \lambda \left(\Phi^a \Phi^a\right)^2. \end{aligned}$$

This can be reduced slightly by breaking the kinetic term into space and time.

This theory is symmetric under $\Lambda \in \mathrm{SO}(n)$,

$$\Phi^a \to \Lambda^{ab} \Phi^b$$

or, infinitesimally,

$$\Phi^a \to \Phi^a + \epsilon^{ab} \Phi^b,$$

where

$$\Lambda^T \Lambda = \mathbb{1}, \qquad \Lambda = \mathbb{1} + \epsilon, \qquad \epsilon^{ab} = -\epsilon^{ba},$$

and we will find the corresponding Noether currents J^μ_{ab}. An infinitesimal transformation $\Phi \to \Phi + \alpha \Delta \Phi$ is a symmetry if the action is invariant up to a surface term, so

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu$$

for some $\mathcal{J}^\mu$. Looking at the deformation of the Lagrangian,

$$\begin{aligned}\alpha\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\Phi}(\alpha\Delta\Phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\partial_\mu(\alpha\Delta\Phi) \\ &= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Delta\Phi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\right]\Delta\Phi, \qquad (P\&S\ 2.11)\end{aligned}$$

and if the field is on-shell, the second term vanishes. Demanding that the remaining term be $\alpha\partial_\mu\mathcal{J}^\mu$ yields

$$\partial_\mu J^\mu = 0, \qquad J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Delta\Phi - \mathcal{J}^\mu.$$

Unsupressing indices,

$$\begin{aligned}J^\mu_{ab} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\epsilon^{ij}_{ab}\Phi_j - \mathcal{J}^\mu_{ab} \\ &= \partial^\mu\Phi_i\,\epsilon^{ij}_{ab}\Phi_j - \mathcal{J}^\mu_{ab},\end{aligned}$$

where the matrices ϵ_{ab} are the generators of the Lie algebra $\mathfrak{so}(n)$, defined as

$$\epsilon^{ij}_{ab} = \begin{cases} 1 & \text{if } (a,b) = (i,j) \\ -1 & \text{if } (a,b) = (j,i) \\ 0 & \text{otherwise}\end{cases}$$

where i, j are obviously over $\mathfrak{so}(n)$, not space.

We are free to let $\mathcal{J}^\mu \to 0^\mu$, giving

$$J^\mu_{ab} = \partial^\mu\Phi_i\,\epsilon^{ij}_{ab}\Phi_j.$$

To check that this current is conserved,

$$\begin{aligned}\partial_\mu J^\mu_{ab} &= \partial_\mu\left[\partial^\mu\Phi_i\,\epsilon^{ij}_{ab}\Phi_j\right] \\ &= \Box\Phi_i\,\epsilon^{ij}_{ab}\Phi_j + \partial^\mu\Phi_i\partial_\mu\epsilon^{ij}_{ab}\Phi_j \\ &= -(m^2 + \lambda\Phi_c\Phi_c)\Phi_i\epsilon^{ij}_{ab}\Phi_j + \epsilon^{ij}_{ab}\partial^\mu\Phi_i\partial_\mu\Phi_j \\ &= 0 + 0 = 0\end{aligned}$$

by symmetry, so J^μ_{ab} is conserved when Φ_a obeys its equations of motion.

The Noether charge is

$$Q_{ab} = \int_{\Omega^3}\mathrm{d}^3x\, J^0_{ab},$$

where Ω^3 is all space and Q is constant in time. We can express J^0 in terms of π and Φ as

$$J^0_{ab} = \partial^0\Phi_i\,\epsilon^{ij}_{ab}\Phi_j = \pi_i\epsilon^{ij}_{ab}\Phi_j.$$

From the definition of ϵ_{ab}, the Noether charge becomes

$$\begin{aligned} Q_{ab} &= \int \mathrm{d}^3x J^0_{ab} \\ &= \int \mathrm{d}^3x\, \pi_i \epsilon^{ij}_{ab} \Phi_j \\ &= \frac{1}{2} \int \mathrm{d}^3x\, \left(\pi_a \Phi_b - \pi_b \Phi_a\right). \end{aligned}$$

When summing over SO(n) indices, we pick up a factor of $\frac{1}{2}$ since the skew-symmetry of ϵ causes us to double-count.

The Noether charges Q_{ab} generate the SO(n) transformations on fields under the Poisson bracket

$$\delta\Phi^a = \left\{\epsilon^{bc} Q_{bc}, \Phi^a\right\}_{\mathrm{P}} = \epsilon^{ab}\Phi^b.$$

To see this, we need the first term in the bracket:

$$\begin{aligned} \epsilon^{bc} Q_{bc} &= \frac{1}{2} \int \mathrm{d}^3x \,\left(\epsilon^{bc}\pi_b \Phi_c - \epsilon^{bc}\pi_c\Phi_b\right) \\ &= \frac{1}{2} \int \mathrm{d}^3x \,\left(\epsilon^{bc}\pi_b \Phi_c - \epsilon^{cb}\pi_b\Phi_c\right) \\ &= \frac{1}{2} \int \mathrm{d}^3x \,\left(\epsilon^{bc}\pi_b \Phi_c + \epsilon^{bc}\pi_b\Phi_c\right) \\ &= \int \mathrm{d}^3x \, \epsilon^{bc}\pi_b \Phi_c. \end{aligned}$$

Now,

$$\begin{aligned} &\left\{\epsilon^{bc} Q_{bc}, \Phi^a\right\}_{\mathrm{P}} = \left\{\int \mathrm{d}^3x'\, \epsilon^{bc}\pi_b(x')\Phi_c(x'), \Phi^a(x)\right\}_{\mathrm{P}} \\ &= \sum_d \left\{ \left(\frac{\partial}{\partial \Phi^d} \int \mathrm{d}^3x'\, \epsilon^{bc}\pi_b\Phi_c\right)\left(\frac{\partial\Phi^a}{\partial\pi^d}\right) - \left(\frac{\partial}{\partial \pi^d} \int \mathrm{d}^3x'\, \epsilon^{bc}\pi_b\Phi_c\right)\left(\frac{\partial\Phi^a}{\partial\Phi^d}\right)\right\} \\ &= \sum_d \left\{ \left(\int \mathrm{d}^3x'\epsilon^{bc}\frac{\partial \pi_b}{\partial\Phi^d}\Phi_c + \int \mathrm{d}^3x'\epsilon^{bc}\pi_b\frac{\partial\Phi_c}{\partial\Phi^d}\right)\left(\frac{\partial\Phi^a}{\partial\pi^d}\right)\right. \\ &\qquad\qquad \left. - \left(\int \mathrm{d}^3x'\epsilon^{bc}\frac{\partial \pi_b}{\partial\pi^d}\Phi_c + \int \mathrm{d}^3x'\epsilon^{bc}\pi_b\frac{\partial\Phi_c}{\partial\pi^d}\right)\delta^{ad}\right\} \\ &= \sum_d \left\{-\int \mathrm{d}^3x\, \epsilon^{bc}\delta_{bd}\delta^{(3)}(x-x')\Phi_c(x')\delta^{ad}\right\} \\ &= -\epsilon^{ac}\Phi_c. \end{aligned}$$

There appears to be a sign error ☹. However, if we define our Poisson bracket backwards, we get $\{\epsilon^{bc}Q_{bc}, \Phi^a\}_{\mathrm{P}} = +\,\epsilon^{ac}\Phi_c$. Then $\delta\Phi^a$ is a generator of an infinitesimal "rotation" in Φ-space.

⇝ Corrections to `fionnf@maths.tcd.ie`.