Furry's Theorem

In QED,

$$\langle\Omega|T[j^\mu(x_1)j^\nu(x_2)j^\lambda(x_3)]|\Omega\rangle = 0$$

where the current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

This is true for any odd number of currents. In other words, the vacuum expectation value of any odd number of electromagnetic currents must vanish.

Charge conjugation C is a unitary linear operator which is conventionally defined to bring a fermion into an antifermion with the same spin. It is a symmetry of QED, so

$$C|\Omega\rangle = |\Omega\rangle\,, \qquad \langle\Omega|C^\dagger = \langle\Omega|.$$

However $j^\mu(x)$ changes sign under C,

We should probably show this, using $\psi \to -i(\bar{\psi}\gamma^0\gamma^2)^T$ and $\bar{\psi} \to -i(\gamma^0\gamma^2\psi)^T$.

$$j^\mu(x) \xrightarrow{C} Cj^\mu(x)C^\dagger = -j^\mu(x),$$

so its vacuum expectation must vanish, as

$$\begin{aligned}\langle\Omega|Tj^\mu(x)|\Omega\rangle &= \langle\Omega|C^\dagger Cj^\mu(x)C^\dagger C|\Omega\rangle \\ &= -\langle\Omega|C^\dagger Tj^\mu(x)C|\Omega\rangle \\ &= -\langle\Omega|Tj^\mu(x)|\Omega\rangle \\ &= 0.\end{aligned}$$

If we now consider an odd number of currents,

$$\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_{2n+1}}(x_{2n+1})]|\Omega\rangle\,,$$

and inserting $C^\dagger C = \mathbb{1}$ as before yields

$$\begin{aligned}&\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_{2n+1}}(x_{2n+1})]|\Omega\rangle \\ &\qquad = \langle\Omega|T[C^\dagger Cj^{\mu_1}(x_1)C^\dagger C\cdots C^\dagger Cj^{\mu_{2n+1}}(x_{2n+1})C^\dagger C]|\Omega\rangle\end{aligned}$$

$$= (-1)^{2n+1}\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_{2n+1}}(x_{2n+1})]|\Omega\rangle$$
$$= -\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_{2n+1}}(x_{2n+1})]|\Omega\rangle$$

since $2n+1$ is odd. Therefore,

$$\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_{2n+1}}(x_{2n+1})]|\Omega\rangle = 0$$

or

$$\langle\Omega|T[j^{\mu_1}(x_1)\cdots j^{\mu_n}(x_n)]|\Omega\rangle = 0$$

for any odd n.

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