Lorentz Covariance in the Coulomb Gauge

In QED, the Coulomb gauge is specified by $\partial_i A^i = 0, A^0 = 0$ for $A^\mu$ the Maxwell potential. Under a Lorentz transformation $\Lambda$ with infinitesimal parameter $\epsilon$,

$$U(\epsilon)A^\mu(x)U^{-1}(\epsilon) = A^\mu(x) - \epsilon^{\mu\nu}A_\nu(x) + \frac{\partial}{\partial x^\mu}\lambda(x, \epsilon)$$

for some operator gauge function $\lambda(x, \epsilon)$.

To see this, we first note that for $A(x)$ a solution of $\partial_\mu F^{\mu\nu} = 0$, there exists $A'(x')$ a solution to the Lorentz transformed system. However, $A^\mu$ does not quite transform as a 4-vector under a Lorentz transformation and we must also consider a supplementary gauge term which admits the $\partial^\mu\lambda(x, \epsilon)$.

$A^\mu(x)$ is an operator in some Hilbert space $\mathcal{H}$ with a unitary representation $U$ of the Lorentz group. Thus,

$$A^\mu(x) \xrightarrow{\Lambda} A'^\mu(x') = U(\epsilon)A^\mu(x)U^{-1}(\epsilon).$$

So,

$$A \rightarrow A' = (1 - \epsilon)A + [\text{gauge terms}],$$

$$A^\mu \rightarrow A'^\mu(x') = A^\mu(x) - \epsilon^{\mu\nu}A'_\nu(x) + \partial^\mu\lambda(x, \epsilon)$$

and similarly,

$$x \rightarrow x' = (1 - \epsilon)x.$$ 

Note that by Lorentz invariance of the metric $g^{\mu\nu}$, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ is an antisymmetric tensor. This is because since we demand invariance, $g = g' \Rightarrow \epsilon g = 0$ necessarily. This determines the antisymmetric form of $\epsilon$ from the symmetry of $g$.

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*Bjorken & Drell* associate primed fields $A'$ with $x$ and unprimed fields $A$ with $x'$. This is counter-intuitive, so I’ve reversed the notation. This explains the sign differences in, for example, *B&J* 11.51.
Any field $\phi$ transforms under a Lorentz transformation $\Lambda$ as

$$\phi(x) \to \phi'(x') = \rho(\Lambda)\phi(\Lambda^{-1}x').$$

(‡)

We are therefore interested in the inverse transformation for $x$.

$$(1 - \epsilon)^{-1} = \sum_{k=0}^{\infty} \epsilon^k = 1 + \epsilon + \epsilon^2 + \cdots$$

$$= 1 + \epsilon + O(\epsilon^2) \approx 1 + \epsilon.$$  

∴ $x = (1 + \epsilon)x'$.  

In component form, $x^\kappa = x'^\kappa + \epsilon^\kappa \sigma x'^\sigma$.

As $A^0 = 0$, its Lorentz transformation must also be zero. (Impossible to boost into a non-zero reference frame, etc.) Taking this transformation,

$$A'^0(x') = A^0(x) - \epsilon^0 \nu A^\nu(x) + \partial^0 \lambda(x, \epsilon) \frac{1}{\epsilon} = 0.$$  

This implies

$$\partial^0 \lambda = \epsilon^0 \nu A^\nu = \epsilon^0 i A^i.$$  

Looking at the Lorentz transformation of $\partial_i A^i$,

$$\partial'_i A'^i(x') = \partial'_i \left[ A^i(x) - \epsilon^i \nu A^\nu(x) + \partial^i \lambda \right].$$  

(♥)

Recall that $\frac{d}{dx^\kappa} = \frac{d}{dx'^\kappa} \frac{dx'^\kappa}{dx^\kappa}$, so we must find the Jacobian terms.†

$$\frac{\partial}{\partial x'^i} x^\kappa = \frac{\partial}{\partial x'^i} \left[ x'^\kappa + \epsilon^\kappa \sigma x'^\sigma \right]$$

$$= \frac{\partial x'^\kappa}{\partial x^\kappa} + \epsilon^\kappa \sigma \frac{\partial x'^\sigma}{\partial x^\kappa}$$

$$= \delta^\kappa_i + \epsilon^\kappa \sigma \delta^\sigma_i.$$  

We can now write ♥ as

$$\partial'_i A'^i(x') = \partial_i \left[ A^i(x) - \epsilon^i \nu A^\nu(x) + \partial^i \lambda \right] \left( \delta^\kappa_i + \epsilon^\kappa \sigma \delta^\sigma_i \right)$$

$$= \delta^\kappa_i \partial_i A^i(x) - \delta^\kappa_i \epsilon^i \nu \partial_i A^\nu(x) + \delta^\kappa_i \partial_i \partial^i \lambda$$

$$+ \epsilon^\kappa \sigma \delta^\sigma_i \partial_i A^i(x) - O(\epsilon^2) + \epsilon^\kappa \sigma \delta^\sigma_i \partial_i \partial^i \lambda$$

$$= \partial_i A^i(x) - \epsilon^i \nu \partial_i A^\nu(x) + \partial_i \partial^i \lambda + \epsilon^\kappa \sigma \delta^\sigma_i \partial_i A^i(x) + \epsilon^\kappa \sigma \partial_i \partial^i \lambda.$$  

Introducing some awful notation, we’ll define the difference between the Lorentz transformed $\partial_i A^i$ and the original by

†This is kind of weird. We can make sense of it by not thinking about (‡) and noticing that $\Lambda : A \to A' = (1 - \epsilon)A + \cdots$, so the argument of $A$ must invert this, $x' \to x = (1 + \epsilon)x'$.  

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\[ \Delta_{\partial A} \equiv \partial'_i A''(x') - \partial_i A'(x) \]
\[ = -\epsilon^i j \partial_j A''(x') + \partial'_i \partial'\lambda + \epsilon^i j \partial_j A'(x) + e^\nu i \partial\nu \partial^i \lambda \]
\[ = -\epsilon^i j \partial_j A''(x') + \epsilon^i j \partial_j A'(x) - \Delta\lambda \]
\[ = \epsilon^i j \partial_j A'(x) - \epsilon^i j \partial_i A''(x') - \Delta\lambda \]

by the definition of the Laplacian \( \Delta \equiv -\partial_i \partial^i \). Decomposing \( \kappa, \nu \) into space and time,

\[ = \epsilon^0 i \partial_0 A'(x) + \epsilon^0 j \partial_j A'(x) - \epsilon^0 j \partial_0 A'(x) - \epsilon^j i \partial_i A'(x) - \Delta\lambda \]
\[ = \epsilon^0 i \partial_0 A'(x) + \epsilon^0 j \partial_j A'(x) - \epsilon^0 j \partial_i A'(x) - \Delta\lambda. \]

Examining the \( \mathcal{O} \)-term,

\[ \mathcal{O} = \epsilon^i j \partial_j A_i - \epsilon^i j \partial_i A_j \]
\[ = \epsilon^i j \left( \partial_j A_i + \partial_i A_j \right) = 0 \]

by symmetry. As \( \Delta_{\partial A} = 0 \), this implies

\[ \epsilon^0 i \partial_0 A' = \Delta\lambda. \quad (\mathcal{O}) \]

Choose \( \Delta\lambda(x, \epsilon) = \phi(x, \epsilon) \). Then \( \phi = \epsilon^0 i \partial_0 A'(x) \) and

\[ \lambda = \int d^3 \bar{x} G(x - \bar{x}) \phi(\bar{x}) \]

where \( G \) is the Green function such that \( \Delta G(x) \equiv \delta^{(3)}(x) \). Several Fourier transformations later (cf. Nigel Buttimore’s CED course), we arrive at the rather Coulombic result

\[ G(x) = -\frac{1}{4\pi r}, \quad r = |x| = \sqrt{x^2} \]

and \( G \) is independent of \( t \). By \( \mathcal{O} \),

\[ \lambda(t, x) = -\int d^3 \bar{x} \frac{\epsilon^0 i \partial_0 A'(t, \bar{x})}{4\pi|x - \bar{x}|}. \]

\footnote{Corrections to fionnf@maths.tcd.ie.}