Homotopic Proof of Cauchy's Theorem

Let

$$\gamma_1, \gamma_2 : [a,b] \to \Omega \subset \mathbb{C}$$
$$\gamma_1(a) = \gamma_2(a), \quad \gamma_1(b) = \gamma_2(b)$$

be piecewise smooth paths. For $t \in [a,b], s \in [0,1]$, assume there exists a homotopy $\Phi(t,s)$ between γ_1, γ_2 relative to $\{a,b\}$:

$$\Phi(t,0) = \gamma_1(t) \quad \Phi(t,1) = \gamma_2(t)$$
$$\Phi(a,s) = \gamma_1(a) = \gamma_2(a)$$
$$\Phi(b,s) = \gamma_1(b) = \gamma_2(b).$$

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then

$$\int_{\gamma_1} f \,\mathrm{d}z = \int_{\gamma_2} f \,\mathrm{d}z.$$

Proof: $\Phi : [a,b] \times [0,1] \to \Omega$ and $\forall (t_0, s_0), \Phi(t_0, s_0) \in \Omega$. Since Ω is open, this implies that for $\varepsilon > 0, \triangle_\varepsilon(\Phi(t_0,s_0)) \subset \Omega$, $\exists \delta > 0$ such that

$$\Phi\left(\triangle_\delta(t_0,s_0)\right) \subset \triangle_\varepsilon\left(\Phi(t_0,s_0)\right).$$

Therefore, there exists a covering of $[a,b] \times [0,1]$ by $\triangle_{\frac{\delta}{2}}(t_0,s_0)$. $[a,b] \times [0,1]$ is compact, so there exists a finite subcover of $\triangle_{\frac{\delta}{2}}(t_0,s_0)$,

$$[a,b] \times [0,1] = \bigcup_k^N \triangle_{\frac{\delta_k}{2}}(t_k, s_k),$$

where

$$a = t_0 < \cdots < t_N = b, \qquad 0 = s_0 < \cdots < s_N = 1,$$

are chosen such that the distance

$$\sqrt{(t_k - t_{k-1})^2 + (s_l - s_{l-1})^2} < \min\left\{\frac{\delta_k}{2}\right\}.$$

Consider the region $[t_{k-1}, t_k] \times [s_{l-1}, s_l]$ and the point $\alpha = (t_k, s_l)$, $\alpha \in \triangle_{\frac{\delta_\mu}{2}}(t^\mu, s^\mu)$. Then $\forall \beta \in [t_{k-1}, t_k] \times [s_{l-1}, s_l]$,

$$\|(t^\mu, s^\mu) - \beta\| \leq \| \underbrace{(t^\mu, s^\mu) - \alpha}_{\leq \frac{\delta_\mu}{2}} \| + \| \underbrace{\alpha - \beta}_{\leq \frac{\delta_\mu}{2}} \| < \delta_\mu.$$

This implies that

$$\beta \in \triangle_{\delta_\mu}(t^\mu, s^\mu),$$

so

$$[t_{k-1}, t_k] \times [s_{l-1}, s_l] \subset \triangle_{\delta_\mu}(t^\mu, s^\mu),$$
$$\Phi\Big([t_{k-1}, t_k] \times [s_{l-1}, s_l]\Big) \subset \triangle_{\varepsilon_\mu}\Big(\Phi(t^\mu, s^\mu)\Big).$$

Since a disc is star-shaped, there exists an antiderivative F of f. Thus, for

$$\tilde{\gamma}_1(t) = \Phi(t, s_{l-1}),$$
$$\tilde{\gamma}_2(t) = \Phi(t, s_l),$$

we conclude that

$$\int_{\tilde{\gamma}_1} f \, \mathrm{d}z = \int_{\tilde{\gamma}_2} f \, \mathrm{d}z,$$

so for

$$\gamma_i = \sum_{\text{all } \tilde{\gamma}\text{s}} \tilde{\gamma}_i,$$

we have

$$\int_{\gamma_2} \mathrm{d}z = \left[\int_{\gamma_1} \mathrm{d}z + \int_{\gamma'} \mathrm{d}z + \int_{\gamma''} \mathrm{d}z\right].$$

This gives

$$\int_{\gamma_2} f \, \mathrm{d}z = \int_{\gamma_1} f \, \mathrm{d}z$$

and proves Cauchy's theorem. □

☾ Corrections to `fionnf@maths.tcd.ie`.