Homological Proof of Cauchy's Theorem

For $\Omega \subset \mathbb{C}, f \in \mathcal{O}(\Omega)$,

$$\int_c f \, \mathrm{d}z = 0$$

where $c = \partial\tilde{c}$ is a 1-cycle.

Proof: for a 2-chain $\tilde{c} = \sum n_j[\varphi_j]$, $\varphi_j : \triangle \to \Omega$,

$$\partial\tilde{c} = \partial \sum n_j[\varphi_j] = \sum n_j \partial[\varphi_j],$$
$$\partial[\varphi_j] = [\varphi \circ \gamma_1] + [\varphi \circ \gamma_2] + [\varphi \circ \gamma_3].$$

For

$$[\gamma] = [\gamma_1] + [\gamma_2] + [\gamma_3],$$

γ is the concatenation of $\gamma_1, \gamma_2, \gamma_3$, $\gamma : [a, b] \to \mathbb{C}$.

Let $a \in \triangle$ be any point in $\triangle$. Then we can construct

$$\tilde{\gamma}(t, \tau) = \left\{ \begin{array}{ll} a & \text{if } \tau = 0 \\ \gamma(t) & \text{if } \tau = 1 \end{array} \right\} = (1 - \tau)a + \tau\gamma(t),$$

a homotopy from $\partial\triangle$ to a. So $\varphi_j\Big(\tilde{\gamma}(t, \tau)\Big)$ is a homotopy from $\partial[\varphi_j]$ to $\varphi_k(a)$. This implies that $\partial[\varphi_j]$ is null homotopic. By the homotopic version of Cauchy's theorem,

$$\int_{\partial[\varphi_j]} f \, \mathrm{d}z = 0.$$

Therefore,

$$\sum n_j \int_{\partial[\varphi_j]} f \, \mathrm{d}z = 0 \quad \Rightarrow \quad \int_c f \, \mathrm{d}z = 0,$$

proving Cauchy's theorem. $\square$

☽ Corrections to `fionnf@maths.tcd.ie`.