CLASSICAL FIELD THEORY
ASSIGNMENT

by
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MA3431: CLASSICAL FIELD THEORY
READING WEEK EXERCISES

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1a,b We have a Lagrangian density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\lambda \phi \, \partial^\lambda \phi + \frac{1}{3} \sigma \phi^3.$$

The Euler-Lagrange equations of motion of a scalar field $\phi(x)$ are

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Therefore, for the above Lagrangian density we have

$$\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} = \frac{\partial}{\partial(\partial_\mu \phi)} \left[\frac{1}{2} \partial_\lambda \phi \, \partial^\lambda \phi \right]$$

as $\frac{\partial}{\partial(\partial_\mu \phi)} \left[\frac{1}{3} \sigma \phi^3 \right] = 0$. So,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2} \frac{\partial(\partial_\lambda \phi)}{\partial(\partial_\mu \phi)} \partial^\lambda \phi + \frac{1}{2} \frac{\partial(\partial^\lambda \phi)}{\partial(\partial_\mu \phi)} \partial_\lambda \phi$$

$$= \frac{1}{2} \frac{\partial(\partial_\lambda \phi)}{\partial(\partial_\mu \phi)} \partial^\lambda \phi + \frac{1}{2} \frac{\partial(\partial_\lambda \phi)}{\partial(\partial_\mu \phi)} \partial^\lambda \phi$$

$$= \frac{1}{2} \delta^\mu_\lambda \partial^\lambda \phi + \frac{1}{2} \delta^\mu_\lambda \partial^\lambda \phi = \partial^\mu \phi$$

and therefore,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = \partial_\mu \partial^\mu \phi = \Box \phi,$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \sigma \phi^2$$ so the E-L gives

$$\Box \phi - \sigma \phi^2 = 0.$$

c

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$$= \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\left[\frac{1}{2}\partial_\lambda \phi \partial^\lambda \phi + \frac{1}{3}\sigma \phi^3\right].$$

d The 4-divergence of this stress tensor is

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\partial^\mu \phi \partial^\nu \phi) - \partial_\mu g^{\mu\nu}\left[\frac{1}{2}\partial_\lambda \phi \partial^\lambda \phi + \frac{1}{3}\sigma\phi^3\right]$$

$$= (\partial_\mu \partial^\mu \phi)\partial^\nu \phi + (\partial_\mu \partial^\nu \phi)\partial^\mu \phi - g^{\mu\nu}\left[\frac{1}{2}\partial_\mu(\partial_\lambda \phi \partial^\lambda \phi) + \frac{1}{3}\sigma \partial_\mu \phi^3\right]$$

$$= (\partial_\mu \partial^\mu \phi)\partial^\nu \phi + (\partial_\mu \partial^\nu \phi)\partial^\mu \phi - g^{\mu\nu}\left[\frac{1}{2}(\partial_\mu \partial_\lambda \phi)\partial^\lambda \phi + \frac{1}{2}(\partial_\mu \partial^\lambda \phi)\partial_\lambda \phi + \frac{1}{3}\sigma \partial_\mu \phi^3\right]$$

$$= (\partial_\mu \partial^\mu \phi)\partial^\nu \phi + (\partial_\mu \partial^\nu \phi)\partial^\mu \phi - \left[\frac{1}{2}(\partial^\nu \partial_\lambda \phi)\partial^\lambda \phi + \frac{1}{2}(\partial^\nu \partial^\lambda \phi)\partial_\lambda \phi + \frac{1}{3}\sigma \partial^\nu \phi^3\right]$$

$$= (\partial_\mu \partial^\mu \phi)\partial^\nu \phi + (\partial_\mu \partial^\nu \phi)\partial^\mu \phi - \frac{1}{2}(\partial^\nu \partial_\lambda \phi)\partial^\lambda \phi - \frac{1}{2}(\partial^\nu \partial^\lambda \phi)\partial_\lambda \phi - \frac{1}{3}\sigma \partial^\nu \phi^3.$$

e Assuming our scalar field $\phi(x^\mu)$ obeys the equation of motion, i.e.

$$\Box\phi - \sigma\phi^2 = 0,$$

$(\partial^\nu \phi^3 = 3\phi^2 \partial^\nu \phi)$

the stress tensor is conserved, that is $\partial_\mu T^{\mu\nu} = 0$, as

$$\partial_\mu T^{\mu\nu} = (\partial_\mu \partial^\mu \phi)\partial^\nu \phi + (\partial_\mu \partial^\nu \phi)\partial^\mu \phi - \frac{1}{2}\delta^\nu_\lambda \sigma \phi^2 \partial^\lambda \phi - \frac{1}{2}\delta^{\nu\lambda}\sigma\phi^2 \partial_\lambda \phi - \sigma\phi^2 \partial^\nu \phi$$

$$= 0.$$

a We have that the electromagnetic field tensor $F^{\mu\nu}$ transforms as

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}$$

with

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}, \quad \Lambda^{\mu}{}_{\rho} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} = \begin{bmatrix} -\beta\gamma E_1 & -\gamma E_1 & -\gamma(E_2+\beta B_3) & -\gamma(E^3-\beta B^2) \\ E_1\gamma & \beta\gamma E_1 & \gamma(\beta E^2-B^3) & \gamma(\beta E^3+B^2) \\ E_2 & B_3 & 0 & -B^1 \\ E_3 & B_2 & B_1 & 0 \end{bmatrix}.$$

Multiplying again by Λ gives

$$\begin{bmatrix} 0 & -E_1 & -\gamma(E^2+\beta B^3) & -\gamma(E^3+\beta B^2) \\ E_1 & 0 & \gamma(E^2\beta-B^3) & -\gamma(E^3\beta+B^2) \\ \gamma(E^2-\beta B^3) & \gamma(B^3-\beta E^2) & 0 & -B_1 \\ \gamma(E^3-\beta B^2) & \gamma(B^2+\beta E^3) & B^1 & 0 \end{bmatrix} = F'^{\mu\nu}.$$

Thus, the electric and magnetic induction fields transform according to

$$E_1' = E_1 \qquad B_1' = B_1$$

$$E_2' = \gamma E_2 - \gamma\beta B_3 \qquad B_2' = \gamma B_2 + \gamma\beta E_3$$

$$E_3' = \gamma E_3 + \gamma\beta B_2 \qquad B_3' = \gamma B_3 - \gamma\beta E_2.$$

.b From Jackson,

$$E_1' = -\frac{qvt'}{r'^3}, \quad E_2' = \frac{qb}{r'^3}, \quad E_3' = 0$$

$$B_1' = 0, \quad B_2' = 0, \quad B_3' = 0$$

with

$$r' = \sqrt{b^2 + (vt')^2}$$

where b is the distance of closest approach. Thus,

$$E_1' = -\frac{qv\gamma t}{(b^2 + (v\gamma t)^2)^{\frac{3}{2}}}, \quad E_2' = \frac{\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{\frac{3}{2}}}, \quad E_3' = 0$$

as required.

$$B_3 = \gamma(B_3' + \beta E_2'),$$

but $B_3' = 0$ (from Jackson), so

$$B_3 = \beta E_2'$$
$$= \frac{\beta\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{\frac{3}{2}}}.$$

We know $B_1 = B_1'$ and $B_2 = \gamma(B_2' - \beta E_3')$, so $B_1 = B_2 = 0$. This gives us

$$E^1 = -\frac{\gamma qvt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad B^1 = 0$$

$$E^2 = \frac{\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad B^2 = 0$$

$$E^3 = 0 \qquad B^3 = \frac{\beta\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}.$$