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$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{c} J_\mu A^\mu.$$

The Euler-Lagrange equations for a 4-potential A^μ are

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right] - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0.$$

Now,

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = -\frac{1}{c} J_\mu$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{2}\frac{\partial}{\partial(\partial^\mu A^\nu)}\left(g_{\sigma\lambda} g_{\tau\kappa} \partial^\lambda A^\kappa \partial^\sigma A^\tau\right)$$

$$= -\frac{1}{2} g_{\sigma\lambda} g_{\tau\kappa}\left(\delta^\mu_\lambda \delta^\nu_\kappa \partial^\lambda A^\kappa + \delta^\mu_\sigma \delta^\nu_\tau \partial^\sigma A^\tau\right)$$

$$= -\frac{1}{2}\left(g_{\sigma\mu} g_{\tau\nu} \partial^\mu A^\nu + g_{\mu\lambda} g_{\nu\kappa} \partial^\mu A^\nu\right)$$

$$= -\frac{1}{2}\left(2\partial_\mu A_\nu\right) = -\partial_\mu A_\nu.$$

Therefore, the Euler-Lagrange equation of motion is

$$\partial^\mu \partial_\mu A_\nu - \frac{1}{c} J_{\cancel{\mu}\nu} = 0.$$

If we assume the Lorenz gauge condition $\partial_\mu A^\mu = 0$, then

$$\partial^\mu F_{\mu\nu} - \frac{1}{c} J_{\cancel{\mu}\nu} = 0$$

$$\partial^\mu F_{\mu\nu} = \frac{1}{c} J_{\cancel{\mu}\nu},$$

The Maxwell equations of electrodynamics, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, as $\partial^\mu \partial_\nu A_\mu = 0$.

Denoting Fermi's Lagrangian density by $\mathcal{L}_F$,

$$\mathcal{L}_F = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{c} J_\mu A^\mu$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu.$$

The difference is therefore

$$\begin{aligned}
\Delta\mathcal{L} &= \mathcal{L} - \mathcal{L}_F \\
&= \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] - \left[-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu\right] \\
&= \frac{1}{4}\left[-F_{\mu\nu}F^{\mu\nu} + 2(\partial_\mu A_\nu \partial^\mu A^\nu)\right] \\
&= \frac{1}{4}\left[2(\partial_\mu A_\nu \partial^\mu A^\nu) - (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right] \\
&= \frac{1}{4}\left[2(\partial_\mu A_\nu \partial^\mu A^\nu) - \{\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\}\right] \\
&= \frac{1}{4}\left[2(\partial_\mu A_\nu \partial^\mu A^\nu) - (2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu)\right] \\
&= \frac{1}{4}\left[2(\partial_\mu A_\nu \partial^\mu A^\nu) - 2(\partial_\mu A_\nu \partial^\mu A^\nu) + 2(\partial_\mu A_\nu \partial^\nu A^\mu)\right] \\
&= \frac{1}{4}\left[2(\partial_\mu A_\nu \partial^\nu A^\mu\right] = \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \\
&= \frac{1}{2}\left[\partial_\mu(A_\nu \partial^\nu A^\mu) + A_\nu \partial_\mu(\partial^\nu A^\mu)\right].
\end{aligned}$$

From the Lorenz gauge condition $\partial_\mu A^\mu = 0$,

$$\partial_\mu(\partial^\nu A^\mu) = \partial^\nu(\partial_\mu A^\mu) = 0$$

and hence

$$\Delta\mathcal{L} = \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu).$$

Thus, the two Lagrangian densities $\mathcal{L}$ and $\mathcal{L}_F$ differ by a 4-divergence ∂_μ of $\frac{1}{2}A_\nu \partial^\nu A^\mu$.

This 4-divergence $\Delta \mathcal{L}$ does not affect the action.

$$S_{\Delta\mathcal{L}} = \int_{\Omega} \Delta \mathcal{L} \, d^4x$$

$$= \frac{1}{4} \int_{\Omega} \partial_\mu (A_\nu \partial^\nu A^\mu) \, d^4x$$

$$= \frac{1}{4} \int_{\partial\Omega} A_\nu \partial^\nu A^\mu \, d\Sigma_\mu$$

over some spacetime region Ω. As Ω expands to consume all spacetime,

$$\int_{\partial\Omega} A_\nu \partial^\nu A^\mu \, d\Sigma_\mu \longrightarrow 0$$

because a physical A_ν must necessarily go to zero due to its inverse dependence on 4-location.

✓

The Euler-Lagrange equations are likewise insensitive to $\Delta\mathcal{L}$.

$$\partial_\mu \left[\frac{\partial(\Delta\mathcal{L})}{\partial(\partial_\mu A_\nu)} \right] - \frac{\partial(\Delta\mathcal{L})}{\partial A_\nu} = 0.$$

$$\frac{\partial(\Delta\mathcal{L})}{\partial A_\nu} = 0 \qquad \text{(No explicit dependence of } \Delta\mathcal{L} \text{ on } A_\nu\text{).}$$

$$\partial_\mu \left[\frac{\partial(\Delta\mathcal{L})}{\partial(\partial_\mu A_\nu)} \right] = 0$$

$$\begin{aligned}
\frac{\partial(\Delta\mathcal{L})}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\frac{1}{2} \partial_\alpha A_\beta \, \partial^\beta A^\alpha \right) \\
&= \frac{1}{2} \left[\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} \partial^\beta A^\alpha + \partial_\alpha A_\beta \frac{\partial(\partial^\beta A^\alpha)}{\partial(\partial_\mu A_\nu)} \right] \\
&= \frac{1}{2} \left[\delta_{\alpha\mu} \delta_{\beta\nu} \, \partial^\beta A^\alpha + \delta^\beta_\mu \delta^\alpha_\nu \, \partial_\alpha A_\beta \right] \\
&= \frac{1}{2} \left[\partial^\nu A^\mu \cdot 2 \right] = \partial^\nu A^\mu.
\end{aligned}$$

Through the Lorenz gauge condition,

$$\partial_\mu \left[\frac{\partial(\Delta\mathcal{L})}{\partial(\partial_\mu A_\nu)} \right] = \partial_\mu (\partial^\nu A^\mu) = 0$$

as before. Hence, the 4-divergence $\Delta\mathcal{L}$ has no effect on the Euler-Lagrange equations of motion. ✓

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu \,, \quad m = \frac{m_\gamma c}{\hbar}.$$

The stress tensor of our Lagrangian density $\mathcal{L}_{\text{Proca}}$ is thus

$$T^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{Proca}}}{\partial(\partial_\mu A_\sigma)} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L}_{\text{Proca}}$$

$$= -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu$$

$$= g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} - F^{\mu\sigma}\partial_\sigma A^\nu + \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu$$

$$= g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} - \left[F^{\mu\sigma}\partial_\sigma A^\nu + A^\nu \partial_\sigma F^{\mu\sigma} + A^\nu\left(\frac{1}{c} J^\mu - m^2 A^\mu\right)\right]$$
$$+ \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu$$

$$= g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} - A^\nu\left[\frac{1}{c} J^\mu - m^2 A^\mu\right] - \partial_\sigma (F^{\mu\sigma} A^\nu)$$
$$+ \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu.$$

Defining $T_D^{\mu\nu}$ as

$$T_D^{\mu\nu} \equiv \Big\{ g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} - A^\nu\left[\frac{1}{c} J^\mu - m^2 A^\mu\right] + \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta$$
$$- \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu$$
$$- \Big[g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} - A^\nu\left(\frac{1}{c} J^\mu - m^2 A^\mu\right) + \frac{1}{2} g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta$$
$$- \frac{m^2}{2} g^{\mu\nu} A_\alpha A^\alpha + \frac{1}{c} g^{\mu\nu} J_\mu A^\mu - 2\partial_\sigma (F^{\mu\sigma} A^\nu)\Big]\Big\},$$

we can now construct our symmetric energy-momentum stress tensor

$$\Theta^{\mu\nu} = T^{\mu\nu} - T_D^{\mu\nu}$$
$$= \Big\{ g^{\beta\nu} F^{\mu\sigma} F_{\sigma\beta} + g^{\mu\beta} F^{\nu\sigma} F_{\sigma\beta} + g^{\mu\nu} F^{\sigma\beta}\partial_\sigma A_\beta - A^\nu\left(\frac{1}{c} J^\mu - m^2 A^\mu\right)$$
$$- A^\mu\left(\frac{1}{c} J^\nu - m^2 A^\nu\right) + \frac{1}{c} J^\nu A^\mu + \frac{1}{c} J^\mu A^\nu - m^2 g^{\mu\nu} A_\alpha A^\alpha \Big\}$$
$$= \Big\{ g^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + m^2\left(A^\mu A^\nu - \frac{1}{2} g^{\mu\nu} A_\rho A^\rho\right)\Big\}$$

The Procca field obeys the Euler-Lagrange equations

$$\partial_\mu\left[\frac{\partial \mathcal{L}_P}{\partial(\partial_\mu A_\nu)}\right] - \frac{\partial \mathcal{L}_P}{\partial A_\nu} = 0.$$

$$\frac{\partial \mathcal{L}_P}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu - \frac{1}{c}J_\mu A^\mu\right)$$

$$= \frac{\partial}{\partial A_\nu}\left(\frac{m^2}{2}g^{\mu\nu}A_\mu A_\nu - \frac{1}{c}g^{\mu\nu}J_\mu A_\nu\right)$$

$$= m^2 g^{\mu\nu}A_\mu - \frac{1}{c}g^{\mu\nu}J_\mu.$$

$$\frac{\partial \mathcal{L}_P}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right)$$

$$= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)\right]$$

$$= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}\Big[(\partial_\rho A_\sigma)(\partial^\rho A^\sigma) - (\partial_\rho A_\sigma)(\partial^\sigma A^\rho) - (\partial_\sigma A_\rho)(\partial^\rho A^\sigma) + (\partial_\sigma A_\rho)(\partial^\sigma A^\rho)\Big]$$

$$= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[2(\partial_\rho A_\sigma)(\partial^\rho A^\sigma) - 2(\partial_\sigma A_\rho)(\partial^\rho A^\sigma)\right]$$

$$= -\frac{1}{4}\left[2\partial^\mu A^\nu + 2\partial^\mu A^\nu - 2\partial^\nu A^\mu - 2\partial^\nu A^\mu\right]$$

$$= -\frac{1}{4}\left[4\partial^\mu A^\nu - 4\partial^\nu A^\mu\right] = -F^{\mu\nu}$$

$$\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] = -\partial_\mu F^{\mu\nu}.$$

Thus, the Procca equations of motion for the massive field are

$$-\partial_\mu F^{\mu\nu} - m^2 g^{\mu\nu}A_\mu + \frac{1}{c}g^{\mu\nu}J_\mu = 0$$

or

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = \frac{1}{c}J^\nu.$$

The differential conservation law for massless electromagnetic fields states

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} J_\rho F^{\rho\nu}.$$

The 4-divergence of the symmetric stress tensor constructed by the Proca field is

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \partial_\mu \left(g^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + m^2 \left(A^\mu A^\nu - \frac{1}{2} g^{\mu\nu} A_\rho A^\rho \right) \right) \\ &= \partial^\rho (F_{\rho\sigma}) F^{\sigma\nu} + F_{\rho\sigma} \partial^\rho (F^{\sigma\nu}) + \frac{1}{2} F_{\rho\sigma} \partial^\nu (F^{\rho\sigma}) \\ &\quad + m^2 \partial_\mu (A^\mu A^\nu) - \frac{m^2}{2} \partial^\nu (A_\rho A^\rho).\end{aligned}$$

From the equation of motion,

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu - m^2 A^\nu,$$

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \left(\frac{1}{c} J_\sigma - m^2 A_\sigma \right) F^{\sigma\nu} + \overbrace{F_{\rho\sigma} \partial^\rho (F^{\sigma\nu}) + \frac{1}{2} F_{\rho\sigma} \partial^\nu (F^{\rho\sigma})}^{\aleph} \\ &\quad + m^2 \partial_\mu (A^\mu A^\nu) - \frac{m^2}{2} \partial^\nu (A_\rho A^\rho).\end{aligned}$$

From ($\aleph$),

$$F^{\mu\sigma} \partial_\mu F_{\sigma\nu} + \frac{1}{2} F^{\mu\sigma} \partial_\nu (F_{\mu\sigma}) = \frac{1}{2} F_{\rho\sigma} \left[\partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\nu} + \partial^\rho F^{\sigma\nu} \right].$$

From Maxwell,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

so

$$\partial_\nu \left(3 \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) = 0$$

$$6 \left(2 \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} \right) = 0$$

$$\frac{1}{2} F_{\rho\sigma} \left[\partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\nu} + \partial^\rho F^{\sigma\nu} \right] = \frac{1}{2} F_{\rho\sigma} \left[\partial^\sigma F^{\rho\nu} + \partial^\rho F^{\sigma\nu} \right] = 0.$$

This gives us

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \left(\frac{1}{c} J_\sigma - m^2 A_\sigma \right) F^{\sigma\nu} + m^2 \partial_\mu (A^\mu A^\nu) - \frac{1}{2} \partial^\nu (A_\rho A^\rho) \\ &= \frac{1}{c} J_\sigma F^{\sigma\nu} + m^2 \left[\partial_\mu (A^\mu A^\nu) - \frac{1}{2} \partial^\nu (A_\rho A^\rho) - A_\sigma F^{\sigma\nu} \right]\end{aligned}$$

which under the Lorenz gauge simplifies to

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} J_\rho F^{\rho\nu}.$$

The Procca symmetric tensor $\Theta^{\mu\nu}$ has time-time component

$$
\begin{aligned}
2\Theta^{00} &= 2F_{0i}F^{i0} + \tfrac{1}{2}F^{0i}F_{i0} + \tfrac{1}{2}F_{ij}F^{ij} + m^2(A^0A^0 + A_iA^i) \\
&= 2E^iE^i - E^iE^i + \tfrac{1}{2}(-\varepsilon^{ijk}B^k)^2 + m^2(A^0A^0 + \vec{A}\cdot\vec{A}) \\
&= \vec{E}^2 + \vec{B}^2 + m(A^0A^0 + \vec{A}\cdot\vec{A})
\end{aligned}
$$

and space-time components

$$
\begin{aligned}
\Theta^{i0} &= g^{i0}F_{0\sigma}F^{\sigma 0} + g^{ii}F_{ij}F^{j0} + m^2(A^iA^0 - \tfrac{1}{2}g^{i0}A_iA^i) \\
&= g^{ii}F_{ij}F^{j0} + m^2(A^iA^0)
\end{aligned}
$$

as the metric tensor has only a non-zero diagonal.

$$
\begin{aligned}
\Theta^{i0} &= E^j\varepsilon^{ijk}B^k + m^2(A^iA^0) \\
&= (\vec{E}\times\vec{B})^i + m^2A^iA^0.
\end{aligned}