The Reissner–Nordström Metric

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Einstein–Maxwell Theory

The electromagnetic Lagrangian density is defined by

$$\mathcal{L}^{\mathrm{E}} = -\frac{\sqrt{-g}}{8\pi} g^{\mu\nu} g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma}$$

with $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. Varying this with respect to the metric gives

$$\begin{aligned} \frac{\delta \mathcal{L}^{\mathrm{E}}}{\delta g^{\mu\nu}} &= \frac{\sqrt{-g}}{4\pi} \left(g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right) \\ &= \sqrt{-g} E_{\mu\nu} \end{aligned}$$

where

$$E_{\mu\nu} = \frac{1}{4\pi} \left(g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right)$$

is the Maxwell energy-momentum tensor.

Properties of $E_{\mu\nu}$ are:

- $E_{\mu\nu} = E_{\nu\mu}$,
- $E^{\mu}{}_{\mu} = 0$,
- $E^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} J_{\nu}$ (in source-free regions, $E_{\mu\nu}$ is conserved).

The Einstein–Maxwell equations (for source-free regions) are

$$G_{\mu\nu} = 8\pi E_{\mu\nu}$$

or

$$G_{\mu\nu} = 2 g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} - \frac{1}{2} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}$$

with $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ and $F^{\mu\nu}{}_{,\nu} = 0$. (Equivalently, $R_{\mu\nu} = 8\pi E_{\mu\nu}$.)

The Reissner–Nordström Solution

We consider the static, spherically symmetric solution of the Einstein–Maxwell equations.

The line element has a general form

$$\mathrm{d}s^2 = -e^{p(r)}\mathrm{d}t^2 + e^{q(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 \sin^2\theta\,\mathrm{d}\phi^2\right)$$

along with the asymptotic flatness boundary condition

$$\lim_{r\to\infty} p(r) = 0, \qquad \lim_{r\to\infty} q(r) = 0.$$

Static, spherical symmetry implies the magnetic field is zero while the electric field is purely radial.

$$\vec{B} = (B_1, B_2, B_3) = 0, \qquad \vec{E} = (E(r), 0, 0)\,,$$

$$\begin{aligned} F_{(a)(b)} &= F_{\mu\nu}\lambda^{\mu}_{(a)}\lambda^{\nu}_{(b)} \\ &= \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

An appropriate orthonormal tetrad is

$$\begin{aligned} \lambda^{\mu}_{(0)} &= \left(e^{-\frac{1}{2}p(r)}, 0, 0, 0\right), \qquad & \lambda^{\mu}_{(1)} &= \left(0, e^{-\frac{1}{2}q(r)}, 0, 0\right), \\ \lambda^{\mu}_{(2)} &= \left(0, 0, +\tfrac{1}{r}, 0\right), \qquad & \lambda^{\mu}_{(3)} &= \left(0, 0, 0, \tfrac{1}{r}\sin\theta\right). \end{aligned}$$

We have that $F_{(a)(b)} \equiv 0$ except

$$F_{(0)(1)} = -F_{(1)(0)} = F_{01}e^{-\frac{1}{2}p(r)}e^{-\frac{1}{2}q(r)}$$

and hence we have $F_{\mu\nu} \equiv 0$ except

$$F_{01} = -F_{10} = -e^{\frac{1}{2}(p+q)}E(r).$$

Similarly, $F^{\mu\nu} \equiv 0$ except

$$F^{01} = -F^{10} = e^{-\frac{1}{2}(p+q)}E(r).$$

Now, since ${F^{\mu\nu}}_{;\nu} = 0$,

$$\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}F^{\mu\nu}\right) = 0$$

using $\Gamma^{\mu}{}_{\mu\lambda} = \frac{\sqrt{-g}_{,\lambda}}{\sqrt{-g}}$ where $\sqrt{-g} = r^2 \sin\theta \; e^{\frac{1}{2}(p+q)}$. Then

$$\begin{aligned} \frac{\partial}{\partial t}\left(\sqrt{-g}F^{10}\right) + \frac{\partial}{\partial r}\left(\sqrt{-g}F^{01}\right) &= 0 \\ \frac{\partial}{\partial r}\left(r^2 \sin\theta \; e^{\frac{1}{2}(p+q)} e^{-\frac{1}{2}(p+q)} E(r)\right) &= 0 \\ \frac{\partial}{\partial r}\left(r^2 E(r)\right) &= 0 \\ r^2 E(r) = \text{constant} &= Q \\ \Rightarrow E(r) &= \frac{Q}{r^2}, \end{aligned}$$

the Coulomb form if we interpret Q as a charge.

The field equations are

$$G^{\mu}{}_{\nu} = 2g^{\lambda\sigma} F^{\mu}{}_{\lambda} F_{\nu\sigma} - \tfrac{1}{2}\delta^{\mu}_{\nu} F_{\lambda\sigma} F^{\lambda\sigma}.$$

Define $Q_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda}$. Then

$$\begin{aligned} G^{\mu}{}_{\nu} &= 2Q^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}_{\nu} Q^{\sigma}{}_{\sigma}, \\ Q_{\mu\nu} &= F_{\mu 0} F_{\nu}{}^{0} + F_{\mu 1} F_{\nu}{}^{1} \\ &= g^{0\lambda} F_{\mu 0} F_{\nu\lambda} + g^{1\lambda} F_{\mu 1} F_{\nu\lambda} \\ &= -e^{-p(r)} F_{\mu 0} F_{\nu 0} + e^{-q(r)} F_{\mu 1} F_{\nu 1}. \end{aligned} \tag{þ}$$

Therefore $Q_{\mu\nu} \equiv 0$ except

$$\begin{aligned} Q_{11} &= -e^{-p(r)} F_{10} F_{10} = -e^{-q(r)} E^2(r) = \frac{-e^{q} Q^2}{r^4}, \\ Q_{00} &= e^{-q(r)} F_{01} F_{01} = e^{p(r)} E^2(r) = \frac{e^{p} Q^2}{r^4}. \end{aligned}$$

Hence

$$\begin{aligned} Q^{\sigma}{}_{\sigma} &= g^{\sigma\lambda} Q_{\sigma\lambda} \\ &= g^{00} Q_{00} + g^{11} Q_{11} \\ &= -\frac{2Q^2}{r^4}. \end{aligned}$$

The right hand side of (þ) is therefore

$$\begin{pmatrix} -\frac{Q^2}{r^4} & & & \\ & -\frac{Q^2}{r^4} & & \\ & & \frac{Q^2}{r^4} & \\ & & & \frac{Q^2}{r^4} \end{pmatrix}$$

and the left hand side of (þ) is

$$G^0{}_0 = G^t{}_t = -\frac{1}{r^2} + \frac{1}{r^2}e^{-q}\left(1 - r\frac{\partial q}{\partial r}\right),$$
$$G^1{}_1 = G^r{}_r = -\frac{1}{r^2} + \frac{1}{r^2}e^{-q}\left(1 + r\frac{\partial q}{\partial r}\right).$$

The field equations are

$$-\frac{1}{r^2} + \frac{1}{r^2}e^{-q}\left(1 - r\frac{\partial q}{\partial r}\right) = -\frac{Q^2}{r^4}, \qquad (A)$$
$$-\frac{1}{r^2} + \frac{1}{r^2}e^{-q}\left(1 + r\frac{\partial q}{\partial r}\right) = -\frac{Q^2}{r^4}. \qquad (B)$$

$A{-}B$ yields

$$\frac{\partial q}{\partial r} + \frac{\partial p}{\partial r} = 0 \quad \Rightarrow \quad p + q = \text{constant}.$$

Asymptotic flatness implies constant $= 0$, so

$$p + q = 0.$$

From A,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-q}\right) = 1 - \frac{Q^2}{r^2},$$
$$re^{-q} = r + \frac{Q^2}{r} + c,$$
$$e^{-q} = 1 + \frac{c}{r} + \frac{Q^2}{r^2}.$$

To retrieve the Schwarzchild solution in the $Q \to 0$ limit, we must have $c = -2M$.

$$e^{-q} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = e^p.$$

The line-element reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

the Reissner–Nordström metric.

Similar to the Schwarzchild case, the Reissner–Nordström solution also has a black hole interpretation. The metric is singular at $r = 0$ and at

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 \qquad \left(\Leftrightarrow \quad r^2 - 2Mr + Q^2 = 0\right)$$
$$r = r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

The solution looks very different depending on the sign of $M^2 - Q^2$.

Case I: $M^2 - Q^2 < 0$

$r^2 - 2Mr + Q^2$ has no real roots and hence there is only a curvature singularity at $r = 0$. The metric is maximal and $r = 0$ is known as a "naked singularity".

> "Nature abhors naked singularities."
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> —*Cosmic censorship conjecture*, Penrose (1969).

Case II: $M^2 - Q^2 > 0$

There are two distinct coordinate singularities at $r = r_\pm$, which may be removed by an appropriate change of coordinates.

The surface $r = r_+$ is very much like the event horizon in the Schwarzchild black hole case (i.e. when an observer at infinity never "sees" a particle crossing the $r = r_+$ surface, infinite red-shift, etc.).

In the region $0 < r < r_-$, the observer need not reach the singularity $r = 0$. In fact, we can cross the $r = r_-$ surface again if we like. In this case, r takes on the nature of a time coordinate again but with the opposite orientation and the observer is forced to move toward the $r = r_+$ surface. He gets spit out of the $r = r_+$ surface (much like emerging from a white hole) into a copy of the universe he started in.

Case III: $M^2 = Q^2$

Two horizons coincide,

$$r = r_+ = r_- = M.$$

This is an extreme Reissner–Nordström black hole (very unstable). In this case, the coordinate r is never timelike in the region $0 < r < M$. We can avoid the singularity and in fact pass through the $r = M$ surface into another copy of the universe we started in.