# Complex Dynamics

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#### Abstract

The aim of this paper is to present some of the main ideas, problems and results relevant to complex dynamics, most of which have been developed over the last 150 years or so. The first task is to define the Fatou and Julia sets which form the foundation of the paper, and develop both local and global theory of dynamical systems interspersed with any polynomial examples which I deem particularly insightful. Of particular interest will be the application of said theory to understanding the classic Newton-Raphson method.

We'll analyse the structure of the Julia and Fatou sets and demonstrate some of the most important theorems and mathematicians connected with the topic. Although this paper is but a brief excursion into the field of complex dynamics, it should provide a firm basis for the study of the numerous more complicated and interesting aspects which have developed from the theory, including a few unsolved conjectures in mathematics. I sincerely hope that any reader extracts as much enjoyment from this paper as I did writing it.

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I dedicate this paper to those who make me smile on a daily basis. You know who you are. Thank you all.

## Chapter 1

# Preface

## 1.1 Introduction

Most of our attention will be focused on holomorphic maps  $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  where  $\hat{\mathbb{C}}$  is the Riemann sphere. This is a representation of the *extended complex numbers*  $\mathbb{C} \cup \{\infty\}$ , which is the complex plane together with a single point at  $\infty$ .

The standard complex plane is mapped bijectively to the sphere by *stereographic projection*. Consider the unit sphere  $S^2$  in  $\mathbb{R}^3$  and think of  $\mathbb{C}$  as the subset plane in  $\mathbb{R}^2$ . We join each point  $z \in \mathbb{C}$  to the point (0,0,1) on the top of the sphere by a straight line and define the stereographic projection of z to be the point on the sphere other than (0,0,1) which is intersected by this line. The extra point at  $\infty$  in the extended plane is considered to be projected to the point (0,0,1) on the sphere.

In  $\hat{\mathbb{C}}$  we interpret  $\frac{1}{0} = \infty$  and  $\frac{1}{\infty} = 0$  which allows us to divide by zero and infinity, but quotients and products involving both 0 and  $\infty$  are left undefined.

Definition 1.1.1 (Rational Functions). Rational functions are functions of the form

$$R(z) = \frac{p(z)}{q(z)}$$

where q(z) is not identically 0 and where the polynomials p(z) and q(z) have no common zeroes. The degree of a rational function is defined as

$$\deg(R) = \max\{\deg(p), \deg(q)\}$$

It follows mainly from the compactness of  $\hat{\mathbb{C}}$  that the holomorphic maps  $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  are either rational functions, or equal to  $\infty$ .

### 1.22 CICRORdal Distance

**Definition 1.2.1** (Chordal Distance). In order to extend the Euclidean metric from  $\mathbb{C}$  to  $\hat{\mathbb{C}}$ , we define the *chordal distance* (straight line distance between the stereographic projections of points on to the Riemann sphere) as follows:

$$\sigma(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}} \qquad z_1, z_2 \in \mathbb{C}$$
  
$$\sigma(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}} \qquad z \in \mathbb{C}$$

 $\sigma$  is also called the *chordal metric* (or *spherical metric*) on  $\hat{\mathbb{C}}$ .

Note also that  $\sigma(0,\infty) = 2$  makes sense when we think of  $\hat{\mathbb{C}}$  as the unit sphere  $S^2$  since the maximum distance between any two points on  $S^2$  is equal to twice the radius. It is also important to see that with this new metric on  $\hat{\mathbb{C}}$  it is possible for a point to be within  $\varepsilon$  of infinity, a concept which isn't in any way innate in standard complex analysis. Most of our analysis will be conducted with respect to this metric on  $\hat{\mathbb{C}}$  unless otherwise stated.

### Notation

We denote iterates of a function f at a point  $z \in \hat{\mathbb{C}}$  in thusly:

$$f^{0}(z) = z$$
  

$$f^{1}(z) = f(z)$$
  

$$f^{2}(z) = f(f(z)) = (f \circ f)(z)$$

and so on. We'll assume at most junctures of this text that n is an integer and n > 0 unless otherwise stated.

**Example 1.2.2** (An Example in  $\mathbb{R}$ ). Take the real-valued polynomial,

$$f(x) = 4x(1-x)$$

with  $x \in \mathbb{R}$ . By completing the square, this can be written as follows:

$$f(x) = 4x(1-x)$$
  
= 4x - 4x<sup>2</sup>  
= -4(x<sup>2</sup> - x)  
= -4(x<sup>2</sup> - x +  $\frac{1}{4} - \frac{1}{4}$ )  
= -4  $\left[ (x - \frac{1}{2})^2 - \frac{1}{4} \right]$   
= 1 - 4  $\left( x - \frac{1}{2} \right)^2$ 

So f(x) is positive in [0, 1] and negative elsewhere. If  $x \in [0, 1]$  initially then the iterates

$$f(x), f^2(x), \dots, f^n(x)$$

stay in [0, 1].

The process of iterating this function is called *cob webbing*. Start with a point  $x \in [0, 1]$  and find f(x). To iterate this new point we bring the point f(x) across to meet the line y = x and then down to intersect the x-axis (effectively just using f(x) to iterate again). Then the process can be repeated. The resulting picture resembles that of a cob web, hence the name.

By inspection, we can see that the web seems to circulate around the point of f(x) where  $x = \frac{3}{4}$ . We'll see later that the point  $x = \frac{3}{4}$  is actually a fixed point for f. These points are a vital key to understanding dynamical systems. This example is at very best a trivial one, so we'll move on to the complex plane (or more specifically the Riemann sphere), where we shall stay for the remainder of the paper.

## Chapter 2

## **Fixed Points**

## 2.1 Fatou and Julia Sets

**Definition 2.1.1** (Equicontinuous). Let  $G \subseteq \hat{\mathbb{C}}$  be open. A family  $\mathcal{F}$  of holomorphic functions  $f: G \to \hat{\mathbb{C}}$  is equicontinuous at  $z_0$  if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that.

$$\sigma(z, z_0) < \delta \qquad \Rightarrow \qquad \sigma(f(z), f(z_0)) < \epsilon$$

for all  $f \in \mathcal{F}$ .  $\mathcal{F}$  is equicontinuous on G if it is equicontinuous at each  $z_0 \in G$ .

The family of functions  $\mathcal{F}$  that we will be focusing on will always be the set of iterates

$$\mathcal{F} = \{ R^n \mid n = 1, 2, \dots \}$$

of a rational function R. Equicontinuity is similar to the definition of continuity of a function except for the critical difference that it now holds for all iterates of the function rather than usually holding for just n = 1. So any points which start off initially very close will stay very close no matter how often we apply R to those points. The main question to address is given a rational function, for which points does this hold and for which does it not. We characterise these two possibilities with the following two sets.

**Definition 2.1.2** (Fatou and Julia Sets). The *Fatou set* of a rational function R is the largest open set  $F(R) \subseteq \hat{\mathbb{C}}$  on which the family of iterates  $R^n$  is equicontinuous for all n > 0. The *Julia set* is the complement of the Fatou Set, denoted J(R).

The Fatou set is the calm or stable set of the function. Points that are initially close to a point in the Fatou set, will stay close. However, in the Julia set, a small change in the initial point  $z_0$  results in a large change to the iterates of  $z_0$  (also known as *sensitive dependence* on initial conditions [11, p 4]). Thus the behaviour of the Julia set is *chaotic*, while the Fatou set is *stable*.

### 2.2. FIXED POINTS **2.2 Fixed Points**

**Definition 2.2.1** (Fixed Points). A point  $z_0$  is called a *fixed point* of p(z) if  $p(z_0) = z_0$ . For a polynomial of degree  $d \ge 2$ , p(z) - z = 0 also has degree d and therefore has d solutions (not necessarily distinct) thus has at most d + 1 fixed points (including  $\infty$ ).

**Example 2.2.2.**  $p(z) = z^2 + z - 4$ . The degree of p is 2, therefore it has at most 3 fixed points. One of these is  $\infty$  because  $p(\infty) = \infty$ . (In fact, we'll prove later that  $\infty$  is always a fixed point for this type of polynomial.) So, there are at most 2 finite fixed points. To compute them let,

$$p(z) = z \iff z^2 + z - 4 = z$$
$$\Leftrightarrow z^2 = 4$$
$$\Leftrightarrow z = \pm 2$$

Therefore the three fixed points for are  $-2, 2, \infty$ .

**Example 2.2.3.**  $p(z) = (z-3)^5 + z$  Here deg p = 5 so it has at most 6 fixed points, counting multiplicities. In fact it only has two. To see this note that  $p(z) - z = (z-3)^5$  so p has one finite fixed point z = 3 with multiplicity 5 and the usual fixed point at  $\infty$ .

**Definition 2.2.4** (Forward Orbit). The forward orbit  $O^+(z_0)$  (or simply orbit) of a point  $z_0$ under a rational mapping R(z) is given by the sequence  $z_0, z_1, z_2, \ldots$  where  $z_n = R(z_{n-1}) = R^n(z_0)$  for  $n = 1, 2, \ldots$ 

Later we'll study quadratics of the form  $p(z) = z^2 + c$ . This map has at most three fixed points, namely,  $\infty$  and \_\_\_\_\_

$$z = \frac{0 \pm \sqrt{-4c}}{2} = \pm i\sqrt{c}$$

What happens to points near these fixed points under iteration? The answer depends on the multiplier  $p'(z_0)$ . Given a fixed point  $z_0$  of p, the linear approximation formula says,

$$p(z) \approx p(z_0) + p'(z_0)(z - z_0) = z_0 + p'(z_0)(z - z_0)$$

for  $z \cong z_0$ . Say  $p'(z_0) = \frac{1}{2}$ . Then  $p(z) \approx \frac{1}{2}(z+z_0)$ , i.e. the distance  $|z-z_0|$  between z and is roughly halved at every iteration. So z gets closer and closer to  $z_0$ . If  $p'(z_0) = 2$  instead say, then  $p(z) \cong 2(z+z_0)$  which means z gets further away from  $z_0$  by a factor of roughly 2 at every iteration of p. This observation motivates the following formal classification of fixed points.

**Definition 2.2.5** (Classification of Fixed Points). A fixed point  $z_0 \in \mathbb{C}$  of p is called:

- 1. attracting if  $0 < |p'(z_0)| < 1$
- 2. superattracting if  $p'(z_0) = 0$
- 3. repelling if  $|p'(z_0)| > 1$

#### 2.3. MÖBIUS TRANSFORMATION

4. *neutral* if  $|p'(z_0)| = 1$ 

We'll deal with the case where  $z_0 = \infty$  later on. To illustrate this better, say p(z) is a polynomial with a fixed point  $z_0$ . Then for z in a neighbourhood of  $z_0$ , we have from the linear approximation formula that,

$$\begin{aligned} |p(z) - z_0| &= |p(z) - p(z_0)| \\ &\approx |p(z_0) + p'(z_0)(z - z_0) - p(z_0)| \\ &= |p'(z_0)(z - z_0)| \\ &= |p'(z_0)| \cdot |z - z_0| \end{aligned}$$

- So if  $z_0$  is attracting then  $|p'(z_0)| < 1$  which implies that  $|p(z) p(z_0)| < |z z_0|$ . Points in a neighbourhood of  $z_0$  get closer and closer at each iteration of p.
- If  $z_0$  is repelling then  $|p'(z_0)| > 1$  which implies  $|p(z) p(z_0)| > |z z_0|$ . The iterates of points in a neighbourhood of  $z_0$  get further and further away from  $z_0$ .
- In the neutral case,  $|p(z) p(z_0)| = |z z_0|$  meaning the iterates stay about the same distance away from  $z_0$  as they were originally and the long term behaviour of such points is less clear.

In fact for an holomorphic function f with an attracting fixed point  $z_0$  such that  $|f'(z_0)| < k < 1$ , we have for z in a neighbourhood of  $z_0$  with  $|z - z_0| < \delta$ ,

$$\begin{aligned} \frac{|f(z) - f(z_0)|}{|z - z_0|} &\leq k \\ \Rightarrow & |f(z) - z_0| &\leq k|z - z_0| < k\delta < \delta \qquad \text{because } |k| < 1 \\ \Rightarrow & |f^n(z) - z_0| &\leq k^n |z - z_0| < k^n \delta < \delta \end{aligned}$$

Then  $f^n(z) \to z_0$  as  $n \to \infty$ . For an attracting (or superattracting) fixed point, by above, there exists an open neighbourhood U of  $z_0$  in which  $f^n(z) \to z_0$  as  $n \to \infty$  for all  $z \in U$ . The set of all points whose orbits converge to  $z_0$  is called the *basin of attraction* for  $z_0$ .

### 2.3 Möbius Transformation

**Definition 2.3.1** (Möbius Transformation). A *Möbius transformation* is a rational map of the form,

$$\phi(z) = \frac{az+b}{cz+d}$$

where  $ad - bc \neq 0$ . In addition, on the Riemann sphere we have  $\phi(\infty) = \frac{a}{c}$  and  $\phi\left(-\frac{d}{b}\right) = \infty$  once  $c \neq 0$ . When c = 0, neither a nor d can be zero, so  $\phi(z)$  is a polynomial and  $\phi(\infty) = \infty$ .

Möbius transformations are rational functions of degree 1, and are bijective holomorphic functions on the Riemann sphere under these additional rules. Any Möbius transformation of the form given above is associated with an  $2 \times 2$  invertible matrix of the form,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where invertibility is guaranteed because,

$$ad - bc \neq 0 \qquad \Rightarrow \qquad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$$

Assume  $\phi(z)$  is a Möbius map with  $c \neq 0$  (i.e. not simply a polynomial) which has a fixed point z then,

$$z = \frac{az+b}{cz+d} \quad \Rightarrow \quad cz^2 + dz = az+b$$

$$\Rightarrow \quad cz^2 + (d-a)z - b = 0$$

$$\Rightarrow \quad z = \frac{(a-d) \pm \sqrt{a^2 + d^2 - 2ad + 4bc}}{2c}$$

$$\Rightarrow \quad z = \frac{(a-d) \pm \sqrt{a^2 + d^2 + 2ad - 4ad + 4bc}}{2c}$$

$$\Rightarrow \quad z = \frac{(a-d) \pm \sqrt{a^2 + d^2 + 2ad - (ad - bc)}}{2c}$$

$$\Rightarrow \quad z = \frac{(a-d) \pm \sqrt{a^2 + d^2 + 2ad - (ad - bc)}}{2c}$$

So  $\phi(z)$  has at most 2 fixed points, possibly only one repeated fixed point. If we think of the Möbius transformations as matrices,

$$\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we can assume that  $det(\phi) = 1$ . This is because changing the variables of the Möbius transformation by a constant  $\lambda \neq 0$  thusly,

$$\frac{az+b}{cz+d} = \frac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)}$$

changes the determinant by a factor of  $\lambda^2$  because,

$$\det \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} = \lambda^2 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So we can normalise and assume that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 1$$

without changing the Möbius transformation. Consider the eigenvalues of the matrix  $\phi$ . These are given by,

$$\det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda -d \end{bmatrix} = 0 \quad \Rightarrow \quad (\lambda - a)(\lambda - d) - bc = 0$$
$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$
$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + 1 = 0$$
$$\Rightarrow \quad \lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4}}{2c}$$

### 2.3.1 Parabolic Case

Say these two eigenvalues are equal. This happens iff  $(a + d)^2 = 4$ , which is known as parabolic Möbius transformation. The matrix  $\phi$  is called parabolic if  $tr^2(\phi) = 4$ . A repeated eigenvalue means that the matrix is conjugate to one of the form,

$$S^{-1}\phi S = \begin{bmatrix} \lambda & 1\\ 0 & \lambda \end{bmatrix}$$

where  $\lambda$  is the single repeated eigenvalue of  $\phi$ . We have assumed that  $\det(\phi) = 1$  so  $\det(S^{-1}\phi S) = 1$  also. Now,

$$det(S^{-1}\phi S) = \lambda^2 = 1 \quad \Rightarrow \quad \lambda^2 = 1$$
$$\Rightarrow \quad S^{-1}\phi S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad S^{-1}\phi S = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

These correspond to the maps  $(S^{-1}\phi S)(z) = z + 1$  and  $(S^{-1}\phi S)(z) = z - 1$  respectively which are both translations whose only fixed point is  $\infty$ .

#### 2.3.2 Non-Parabolic Case

There are two distinct eigenvalues iff  $(a + d)^2 \neq 4$  (i.e. if the Möbius transformation is nonparabolic). Then the matrix is diagonalisable to a matrix,

$$S^{-1}\phi S = \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix}$$

$$det(S^{-1}\phi S) = \lambda_1 \lambda_2 = 1 \quad \Rightarrow \quad \lambda_1 = \frac{1}{\lambda_2}$$
$$\Rightarrow \quad S^{-1}\phi S = \begin{bmatrix} \lambda_1 & 0\\ 0 & \frac{1}{\lambda_1} \end{bmatrix}$$

Write  $\lambda = \lambda_1$  for simplicity, then  $\phi$  is conjugate to a translation/dilation map  $(S^{-1}\phi S)(z) = \lambda^2 z$ . This comes from a Möbius map where  $a = \lambda$ , b = 0, c = 0 and  $d = \frac{1}{\lambda}$ .

## 2.4 Fixed Points of Möbius Transformations

In the following we'll refer to the concept of conjugacy of two dynamical systems which will be formalised later. For now it is sufficient to be familiar with conjugacy of matrices for the purpose of understanding the discussion. This is adapted from [3, p 4, 5]. The cases for fixed points of a Möbius transformation (now called R for lazy typist reasons) are detailed as follows.

If the matrix for the transformation is parabolic then it is conjugate to a translation which we know has only one fixed point at infinity. This gives rise to the first two cases.

### 2.4.1 A single fixed point at $\infty$

In this case c = 0,

$$\begin{array}{ll} \Rightarrow & R(z) = z + \alpha & \text{ for some } \alpha \neq 0 \\ \Rightarrow & R^n(z) = z + n\alpha & \text{ Translation} \\ \Rightarrow & R^n(z) \rightarrow \infty & \text{ as } n \rightarrow \infty \text{ for all } z \end{array}$$

### 2.4.2 A single fixed point in $\mathbb{C}$

Say  $\zeta \in \mathbb{C}$  is a unique fixed point for R(z). Let

$$g(z) = \frac{1}{z - \zeta}$$

which is a Möbius map taking  $\zeta$  to  $\infty$ . Define  $S(z) = (gRg^{-1})(z)$ . Since the composition of Möbius transformations is Möbius, we know that S is a Möbius transformation. In terms of matrices, S and R are conjugate with this choice of S. Note that,

$$(gRg^{-1})(\infty) = gR(\zeta)$$
$$= g(\zeta)$$
$$= \infty$$

$$\begin{split} S^n(z) &= [(gRg^{-1})(gRg^{-1})\cdots(gRg^{-1})](z) \\ &= (gR^ng^{-1})(z) \end{split}$$

Replace z with g(z) and take  $g^{-1}$  of both sides to see that,

$$g^{-1}S^n(g(z)) = R^n(z)$$

where  $S^n(g(z)) \to \infty$  as  $n \to \infty$  for any z. This implies that,

$$R^n(z) \to g^{-1}(\infty) = \zeta$$

as  $n \to \infty$  for any z. Thus all points converge to the fixed point  $\zeta$  under iteration of R. Here we have shown that the Möbius transformation in case 2 is conjugate to one of the form given in case 1. The combination of these two cases means that if R has a unique fixed point then every point in  $\hat{\mathbb{C}}$  converges to that point under R.

If the matrix for the transformation is non-parabolic then it is conjugate to a translation/dilation map which we know fixes only the points at 0 and infinity. This gives rise to the next (and last) two cases.

### 2.4.3 Two distinct fixed points at 0 and $\infty$

In this case, R(z) = kz with  $k \neq 0$  which implies  $R^n(z) = k^n z$ . There are three possibilities for k.

- 1. |k| < 1 which implies that  $R^n(z) \to 0$  as  $n \to \infty$
- 2. |k| = 1 which implies that  $|R^n(z)| = |z|$  as  $n \to \infty$
- 3. |k| > 1 which implies that  $R^n(z) \to \infty$  as  $n \to \infty$

### 2.4.4 Two distinct fixed points in $\mathbb{C}$

Say R has fixed points  $\zeta_1, \zeta_2 \in \mathbb{C}$ , where  $\zeta_1 \neq \zeta_2$ . Construct a Möbius transformation g such that  $\zeta_1 \mapsto 0$  and  $\zeta_2 \mapsto \infty$ . The function

$$g(z) = \frac{z - \zeta_1}{z - \zeta_2}$$

2.4. FIXED POINTS OF MÖBIUS TRANSFORMATIONS CHAPTER 2. FIXED POINTS will suffice. As before, let  $S = gRg^{-1}$ . This fixes the points 0 and  $\infty$  by the following:

$$(gRg^{-1})(0) = gR[g^{-1}(0)]$$
  
=  $gR(\zeta_1)$   
=  $0$ 

because  $\zeta_1$  is a fixed point of R and

$$(gRg^{-1})(\infty) = gR[g^{-1}(\infty)]$$
$$= gR(\zeta_2)$$
$$= g(\zeta_2)$$
$$= \infty$$

because  $\zeta_2$  is a fixed point of R.  $S^n$  is thus conjugate to the mapping described in 3. Applying the previous conclusions, we see that  $R^n(z) = g^{-1}S^ng(z)$  has three cases based on the behaviour of  $S^n$ 

- 1.  $S^n(g(z)) \to 0$  implies that  $R^n(z) \to \zeta_1$
- 2.  $S^n(g(z)) \to \infty$  implies that  $R^n(z) \to \zeta_2$
- 3.  $|S^n(g(z))| \to |g(z)|$  implies that  $|R^n(z)| \to |z|$

where case 3 is so because g maps circles to circles as does  $g^{-1}$ . So the iterates either converge to one of the fixed points, or move around a circle. It is important to note that not all of the fixed points of a Möbius transformation are attracting. Consider the Möbius map  $\phi(z) = az + b$ where  $a, b \neq 0$ . For a fixed point of this map we solve,

$$z = az + b \qquad \Rightarrow \qquad z = \frac{b}{1-a}$$

So if a = 1 the only fixed point is  $\infty$ , but otherwise the fixed point is in  $\mathbb{C}$ . Also  $\phi'(z) = a$ , which implies that z is an attracting fixed point when |a| < 1 but repelling otherwise. It is important to note that all points can eventually converge to one of the fixed points even if they are initially repelled away.

## Chapter 3

# **Basic Iteration**

## 3.1 Polynomials of Higher Degrees

**Definition 3.1.1** (Fixed point at  $\infty$ ). From now on, we will assume both the polynomials p(z) and rational functions R(z) that we deal with are of degree at least 2. For p(z), the fixed point at  $\infty$  is called:

- 1. attracting if 0 < |f'(0)| < 1,
- 2. superattracting if f'(0) = 0,

where

$$f(z) = \frac{1}{p\left(\frac{1}{z}\right)}$$

This definition allows us to prove a claim made earlier.

**Theorem 3.1.2.**  $\infty$  is a superattracting fixed point for any p(z) with degree  $n \ge 2$ .

**Proof.** Say  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$  where  $a_n \neq 0$  and  $n \geq 2$ . Let

$$f(z) = \frac{1}{p\left(\frac{1}{z}\right)}$$

Then,

$$f(z) = \frac{1}{\frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \dots + \frac{a_1}{z} + a_0}$$
$$= \frac{z^n}{a_n + a_{n-1}z + \dots + a_1z^{n-1} + a_0z^n}$$

Now,

$$f(0) = \frac{0^n}{a_n} = 0 \qquad \Leftrightarrow \qquad \frac{1}{p(\infty)} = 0 \qquad \Leftrightarrow \qquad p(\infty) = \infty$$

To show it is superattracting, differentiate f'(z) by using the quotient rule with  $u = z^n$  and

 $v = a_n + a_{n-1}z + \dots + a_1z^{n-1} + a_0z^n$  to get:

$$f'(z) = \frac{nz^{n-1}(a_n + a_{n-1}z + \dots + a_1z^{n-1} + a_0z^n) - z^n(a_{n-1} + 2a_{n-2}z + \dots + a_0z^{n-1})}{(a_n + a_{n-1}z + \dots + a_1z^{n-1} + a_0z^n)^2}$$

whence

$$f'(0) = 0$$

**Definition 3.1.3** (Periodicity). A point  $z_0$  is called a *periodic point* of p(z) (with period n) if  $p^n(z_0) = z_0$  for some integer  $n \ge 1$ . Note that if n = 1 then  $z_0$  is a fixed point and thus trivally periodic.

**Definition 3.1.4** (Pre-Periodicity). A point is called a *pre-periodic point* of p(z) if  $p^{n+j}(z_0) = p^j(z_0)$  for some integer j > 0. In other words, if some iterate  $p^j(z_0)$  of  $z_0$  under p(z) is periodic.

Theorem 3.1.5. Attracting fixed points are in the Fatou set, as are their basins of attraction.

**Proof.** To see this first we'll show that attracting fixed points are in the Fatou set. Suppose first that  $z_0$  is an attracting fixed point. Then for z nearby  $z_0$ , with  $|z - z_0| < \delta$ , we have the following:

$$p(z) = p(z_0) + p'(z_0)(z - z_0) = z_0 + p'(z_0)(z - z_0)$$

Then,

$$|p(z) - p(z_0)| = |p(z) - z_0|$$
  

$$\approx |z_0 + p'(z_0)(z - z_0) - z_0|$$
  

$$= |p'(z_0)(z - z_0)|$$
  

$$= |p'(z_0)| \cdot |z - z_0|$$
  

$$< 1 \cdot \delta$$

So the set of iterates are equicontinuous at  $z_0$  since  $|z_0 - z| < \delta$  implies that  $|p(z) - p(z_0)| < \delta$ . Now since  $p^n(z) \to z_0$  as  $n \to \infty$  for any z in the basin of attraction of  $z_0$ , there exists  $N \in \mathbb{N}$  such that for n > N,  $|p^n(z) - z_0| < \delta$  and thus the above argument holds for z.

**Example 3.1.6.** Let  $p(z) = z^2$ . Then the orbit of any z under p is given by the following:

$$z \mapsto p(z) = z^2 \mapsto p^2(z) = z^4 \mapsto p^3(z) = z^8 \mapsto \cdots$$

and so on. The three points 0, 1 and  $\infty$  are fixed points of this map since p(0) = 0, p(1) = 1and  $p(\infty) = \infty$ . We've proved already that  $\infty$  is always a superattracting fixed point of any polynomial map of degree at least 2. The fixed point at 0 is also superattracting because,

$$p'(z) = 2z \qquad \mapsto \qquad p'(0) = 0$$

However, the fixed point at 1 is repelling because,

$$p'(1) = 2 > 1$$

#### 3.1. POLYNOMIALS OF HIGHER DEGREES

For  $|z_0| < 1$ ,  $p^n(z_0) \to 0$  as  $n \to \infty$ . Thus  $|z_0| < 1$  is the basin of attraction for the attracting fixed point at 0 and so belongs to the Fatou set. For  $|z_0| > 1$ ,  $p^n(z_0) \to \infty$  as  $n \to \infty$ . So  $|z_0| > 1$  is the basin of attraction for the fixed point at  $\infty$ , and also belongs to the Fatou set. Therefore the unit circle

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$$\Delta = \{z : |z| = 1\}$$

is the Julia set of  $p(z) = z^2$ . To see this choose r such that |r| < 1 and consider the point  $z \in \Delta$ . We'll show that even if z and rz are close initially, their iterates exhibit very different behaviour. Assume the opposite, i.e. that z is in the Fatou set (we already know that rz is). Then,

$$\sigma(z, rz) < \delta \qquad \Leftrightarrow \qquad \sigma(p^n(z), p^n(rz)) < \epsilon$$

Choose  $\epsilon = \frac{1}{3}\sigma(0,\infty) = \frac{2}{3}$ . Now we can deal with Euclidean norm because we're working only on, or inside  $\Delta$ . So for  $w \in \overline{\mathbb{D}}$  we have

$$\sigma(0,w) = \frac{2|w|}{\sqrt{1+|w|^2}} \le 2|w|$$

and  $\sigma$  increases as the Euclidean distance does. So then we have:

$$|z - rz| < \delta \quad \Rightarrow \quad |p^n(z) - p^n(rz)| < \frac{2}{3}$$

Also  $p^n(rz) \to 0$  as  $n \to \infty$  because |rz| < 1. So given any  $\eta > 0$ , there exists N such that for n > N,

$$|p^n(rz)| < \eta$$

Set  $\eta = \frac{1}{3}$ . Then,

$$\begin{aligned} |p^{n}(z) - p^{n}(rz)| &\geq ||p^{n}(z)| - |p^{n}(rz)|| & \text{reverse triangle inequality} \\ &> |1 - \eta| & \text{since } |p^{n}(z)| < \eta \text{ and } z \in \Delta \\ &= \left|1 - \frac{1}{3}\right| \\ &= \frac{2}{3} \end{aligned}$$

which is a contradiction, meaning p is not equicontinuous at z. Therefore z isn't in the Fatou set, so it must be in the Julia set of p.  $\Delta$  is both said to be invariant under p: every point on  $\Delta$  is mapped to another point on  $\Delta$  by p. But what happens to the long term iterates of points on  $\Delta$ ? Let  $z = e^{i\theta}$  be a point on  $\Delta$ . Then,

$$p(z) = e^{2i\theta}$$

$$p^{2}(z) = e^{4i\theta}$$

$$p^{3}(z) = e^{8i\theta}$$

$$\vdots$$

$$p^{n}(z) = e^{2^{n}i\theta}$$

#### 3.2. CONJUGACY

#### CHAPTER 3. BASIC ITERATION

The squaring function p doubles the angle of these points at every iteration. Assume n > 0, i.e. there is at least one iteration of the function. We'll ignore the fixed points 0 and  $\infty$  which are trivially periodic but aren't on  $\Delta$ . Then for:

$$\begin{array}{lll} n=1 & p(z)=z^2 & \text{Periodic Points if } z^2=z & \Leftrightarrow & z=1 \\ n=2 & p(z)=z^4 & \text{Periodic Points if } z^4=z & \Leftrightarrow & z^3=1 \\ & \Leftrightarrow & z=1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i} \\ n=3 & p(z)=z^8 & \text{Periodic Points if } z^8=z & \Leftrightarrow & z^7=1 \\ & \Leftrightarrow & z=1, e^{\frac{2\pi}{7}i}, e^{\frac{4\pi}{7}i}, \dots, e^{\frac{12\pi}{7}i} \end{array}$$

So for the  $n^{th}$  iterate  $p^n(z) = z^{2^n}$ , a point is periodic if

$$z^{2^{n}} = z \quad \Leftrightarrow z^{2^{n}-1} = 1$$
$$= z = \exp\left(\frac{2j}{2^{n}-1}\pi i\right)$$

where  $j = 0, 1, 2, ..., 2^n - 2$ . So  $z = e^{i\theta}$  is periodic if  $\theta$  is of the form,

$$\theta = \frac{2j}{2^n - 1}\pi$$

Which is equivalent to saying that  $\theta = \frac{p}{q}\pi$  where p is even and q is odd.

## 3.2 Conjugacy

For quadratic maps, we may restrict our attention to polynomials of the form  $z^2 + c$ . This is due to the fact that any quadratic map is (dynamically) equivalent to one of the form  $p(z) = z^2 + c$ for  $c \in \mathbb{C}$ . In some sense the dynamics of an arbitrary quadratic look the same as those for some  $p(z) = z^2 + c$ . We'll justify this restriction later. First we'll give the definition of conjugacy followed by a discussion of one of the most interesting applications of complex dynamics; approximating the roots of a cubic function (otherwise known as Newton's method).

**Definition 3.2.1** (Conjugacy). Two maps  $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  and  $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  are called *conjugate* if there exists a homeomorphism (continuous bijection which has a continuous inverse)  $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  such that  $h \circ f = g \circ h$ . In other words, the following diagram commutes:



$$f = h^{-1} \circ g \circ h$$
  

$$\Rightarrow f^n = (h^{-1} \circ g \circ h)^n = h^{-1} \circ g \circ \underbrace{h \circ h^{-1}}_{= \operatorname{id}} \circ g \circ \underbrace{h \circ h^{-1}}_{= \operatorname{id}} \circ \cdots = h^{-1} \circ g^n \circ h$$

So, h maps the orbits of f to those of g, and  $h^{-1}$  maps orbits of g to those of f (as well as all their iterates) thus inducing a 1-1 correspondence between the orbits. Furthermore, fixed points are mapped to fixed points and their corresponding multipliers are equal.

The maps f and g can be considered the same maps viewed in different coordinate systems. A change of variables transforms one map into the other. Now we can prove a claim from earlier.

**Theorem 3.2.2.** Let  $f(z) = az^2 + bz + c$  be a quadratic where  $a, b, c \in \mathbb{C}$  with  $a \neq 0$ . Then f is conjugate to some  $p(z) = z^2 + k$  where  $k \in \mathbb{C}$ . [11, p 8]

**Proof.** Need a function h such that  $h \circ f = p \circ h$ . Let h be the affine conjugacy  $h(z) = az + \frac{b}{2}$ . Firstly.

$$(h \circ f)(z) = h(az^2 + bz + c) = a^2z^2 + abz + ac + \frac{b}{2}$$

Also,

$$(p \circ h)(z) = p\left(az + \frac{b}{2}\right) = \left(az + \frac{b}{2}\right)^2 + k = a^2 z^2 + \frac{b^2}{4} + abz + k$$

These two are equal when  $k = ac + \frac{b}{2} - \frac{b^2}{4}$ . Therefore f is conjugate to p for this choice of k.

Example 3.2.3. Perhaps the most important aspect of these maps is what happens to the critical point 0. This is the only critical point of any  $p(z) = z^2 + c$  since

 $p'(z) = 0 \quad \Rightarrow \quad 2z = 0 \quad \Rightarrow \quad z = 0$ 

In the previous two examples,  $O^+(0)$  stayed bounded for all n. We'll show later that if  $O^+(0)$ escapes to  $\infty$  then J is a Cantor set. If not then J is connected [11, p 13]. This naturally leads to the definition of Benoît Mandelbrot's eponymous set.

**Definition 3.2.4** (Mandelbrot set). Given  $p(z) = z^2 + c$ , the Mandelbrot set  $\mathcal{M}$  is defined as the set

$$\mathcal{M} = \{ c \in \mathbb{C} : p^n(0) < \infty \}$$

for all n > 0. It is the set of values of c for which the orbit  $O^+(0)$  of the critical point 0, stays bounded. Equivalently, it is the set of c for which J(p) is connected.

So for  $c \in \mathcal{M}$ , the iterates of 0 remain bounded and therefore J(p) is connected. For  $c \notin \mathcal{M}, J(p)$  is a Cantor set. As we've seen, 0 is an element of the Mandelbrot set since the corresponding polynomial,  $p(z) = z^2$  has connected Julia set (A forthcoming example will show

### CHAPTER 3. BASIC ITERATION

that  $-2 \in \mathcal{M}$  also). However,  $1 \notin \mathcal{M}$ . If we look at  $p(z) = z^2 + 1$  at 0 then we see that:

$$p(z) = z^{2} + 1 \qquad p(0) = 1$$

$$p^{2}(z) = (z^{2} + 1)^{2} + 1 = z^{4} + 2z^{2} + 2 \qquad p^{2}(0) = 2$$

$$p^{3}(z) = (z^{4} + 2z^{2} + 2)^{3} + 1 = z^{8} + 4z^{6} + 8z^{4} + 8z^{2} + 5 \qquad p^{3}(0) = 5$$

So  $O^+(0)$  is a monotonic increasing sequence for this choice of c meaning that  $p^n(0)$  doesn't remain bounded. As one would expect there are also elements of  $\mathbb{C} \setminus \mathbb{R}$  which are in  $\mathcal{M}$ . We'll state without proof, for the purpose of trivial examples, that  $i \in \mathcal{M}$ . This can be easily verified using the above example as a template.

As a more general rule, if we note that the Mandelbrot set is contained in the closed disk of radius 2 around the origin, we can state that  $c \in M$  if  $|p^n(0)| < 2$  for all  $n \in \mathbb{N}$ . From this, we deduce that Julia sets J(p) for large values of c (by this we mean |c| > 2) are always complicated Cantor sets. I hasten to point out that there are points c with  $|c| \leq 2$  which also give cantor sets (c = 1 for instance, as we've just seen  $1 \notin M$ ), but every point in { $c \in \mathbb{C} : |c| > 2$ } has this property.

**Example 3.2.5.** This example can be found (albeit in less detail) in [2, p 29]. Take  $p(z) = z^2 - 2$  which has fixed points at 2, -1 and  $\infty$ . First note that if  $z \in [-2, 2]$  then  $0 \le z^2 \le 4$ , and hence  $-2 \le z^2 - 2 \le 2$ . So points on the line segment [-2, 2] are mapped into [-2, 2] by p and thus is invariant under p.

Consider the conformal map  $h(\zeta) = \zeta + \frac{1}{\zeta}$  of  $\{\zeta \in \mathbb{C} : |\zeta| > 1\}$  onto  $\mathbb{C} \setminus [-2, 2]$ . We'll show that this map is surjective. Firstly if  $z = e^{i\theta}$ ,

$$h(z) = e^{i\theta} + \frac{1}{e^{i\theta}}$$
  
=  $e^{i\theta} + e^{-i\theta}$   
=  $\cos\theta + i\sin\theta + \cos\theta - i\sin\theta$   
=  $2\cos\theta$ 

which is always between -2 and 2 and so is contained in the line segment [-2, 2]. Now  $h(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  and solving h(z) = w means solving:

$$z + \frac{1}{z} = w \qquad \Leftrightarrow \qquad z^2 + 1 = wz \qquad \Leftrightarrow \qquad z^2 - wz + 1 = 0$$

where the rightmost equation has two solutions. If z is one solution then 1/z is the other because  $h(z) = h(\frac{1}{z})$ . So if one solution of h(z) = w is inside the unit circle  $\Delta$ , then the other is outside because,

$$|z| < 1 \qquad \Leftrightarrow \qquad \frac{1}{|z|} > 1$$

If  $|z| \neq 1$  (z not on  $\Delta$ ) then  $h(z) = w \notin [-2, 2]$ . Hence, there exists a solution with |z| > 1.

Therefore  $h : \{z \in \mathbb{C} : |z| > 1\} \to [-2, 2]$  surjectively. We'll now show that  $f(\zeta) = \zeta^2$  is conjugate to p(z) under the mapping h, i.e. that the following diagram commutes:

Check that  $p \circ h = h \circ f$ :

$$(p \circ h)(\zeta) = p\left(\zeta + \frac{1}{\zeta}\right)$$
$$= \left(\zeta + \frac{1}{\zeta}\right)^2 - 2$$
$$= \zeta^2 + \frac{1}{\zeta^2} + 2 - 2$$
$$= h(\zeta^2)$$
$$= (h \circ f)(\zeta)$$

So the dynamics of p(z) on  $\mathbb{C} \setminus [-2, 2]$  are the same as those of  $f(\zeta) = \zeta^2$  on  $\{\zeta \in \mathbb{C} : |\zeta| > 1\}$ . This is the same as the squaring function example earlier, and thus all points in the domain  $\{\zeta \in \mathbb{C} : |\zeta| > 1\}$  tend to infinity under iteration of f. Therefore all points in  $\mathbb{C} \setminus [-2, 2]$  tend to infinity under iteration of p making this the basin of attraction for  $\infty$ . Consequently, for any point in [-2, 2], its orbit stays in [-2, 2] which means the Julia set for  $p(z) = z^2 - 2$  is J(p) = [-2, 2].

To reinforce this claim, note that the orbit of points very near [-2, 2] can tend to infinity while the orbits of points inside [-2, 2] stay bounded thus exhibiting the chaotic (and nonequicontinuous) behaviour expected of the Julia set. It is worth noting for later on, that although this example is also of the form  $p(z) = z^2 + c$ , the Fatou set only has one component in this case, whereas for  $p(z) = z^2$  it had two. This observation will be formalised in a theorem later on.

**Definition 3.2.6** (Misiurewicz Point). Given a polynomial of the form  $p(z) = z^2 + c$ , any value of c for which  $O^+(0)$  is pre-periodic (but not periodic) is called a *Misiurewicz point*.

In the above example where c = -2 the critical orbit is given by  $0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto so 0$  is pre-periodic and thus c = -2 is a Misiurewicz point. This is more of a supplementary definition as we won't encounter it again in this paper. Misiurewicz points are important in the study of the Mandelbrot set, particularly for the study of external rays developed by Douady and Hubbard. For more on this, see [11, p 15]. They are also credited with showing connectivity of the Mandelbrot set which was conjectured but never proven by Mandelbrot himself. Local connectivity of the Mandelbrot set is an open conjecture in Mathematics to this day.

## Chapter 4

# Local Conjugations at Fixed Points

This chapter is based on [2, p 31-33]. While we have explicitly calculated h in the example in the previous chapter, there are some cases in which the conjugation function h might not exist. We'll simplify imminent analysis by using the following notation: denote by  $\lambda$ , the value of the first derivative of the function f at a fixed point  $z_0$ , so  $\lambda = f'(z_0)$ . This is called the *multiplier* of f at  $z_0$ .

## 4.1 Koening's Theorem

**Theorem 4.1.1** (Koenigs). Let  $z_0$  be an attracting fixed point of a rational function f which is not superattracting, i.e.  $0 < |\lambda| < 1$ . Then there exists conformal  $\varphi(z)$  mapping a neighbourhood of  $z_0$  onto a neighbourhood of 0 such that  $\varphi$  conjugates f(z) to a linear function  $g(z) = \lambda z$ . Furthermore, this conjugate function  $\varphi$  is unique up to multiplication by a non-zero scalar.

**Proof.** We already have  $f(z_0) = z_0$  and we can furthermore assume that  $z_0 = 0$  by changing coordinates and translating. The rational function  $f(z) = \frac{p(z)}{q(z)}$  is holomorphic away from its poles, i.e. where q(z) = 0. Because  $f(0) = 0 \neq \infty$ , f does not have a pole there. So there exists some ball of radius r > 0 around zero inside which, q has no zeroes and therefore f has no poles. So f is holomorphic inside the ball with radius  $r = \min\{\zeta : q(\zeta) = 0\}$ . Now because f is holomorphic in the disc of radius r, there exists a power series expansion for f about 0 in that disc:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \cdots$$
  
=  $\lambda z + a_2 z^2 + a_3 z^3 + \cdots$ 

for |z| < r where r > 0. So,

$$(f \circ f)(z) = \lambda f(z) + a_2(f(z))^2 + a_3(f(z))^3 + \cdots = \lambda^2 z + \lambda a_2 z^2 + \lambda a_3 z^3 + \cdots + a_2(\lambda^2 z^2 + 2\lambda a_2 z^3 + \cdots) + a_3(\lambda^3 z^3 + \cdots) + \cdots$$

## 4.1. KOENING'S THEORENCHAPTER 4. LOCAL CONJUGATIONS AT FIXED POINTS Now define $\varphi_n(z) = \frac{f^n(z)}{\lambda^n}$ . The sequence proceeds as follows

$$\varphi_1(z) = \frac{f(z)}{\lambda}$$
  $\varphi_2(z) = \frac{f^2(z)}{\lambda^2}$   $\cdots$   $\varphi_n(z) = \frac{f^n(z)}{\lambda^n}$ 

The idea is that if  $\varphi_n(z) \to \varphi$  as  $n \to \infty$  then,

$$(\varphi_n \circ f)(z) \to \varphi(f(z))$$
 and  $\lambda \varphi_{n+1}(z) \to \lambda \varphi(z)$  so  $\varphi(f(z)) = \lambda \varphi(z)$ 

Indeed we have,

$$(\varphi_n \circ f)(z) = \varphi_n(f(z)) = \frac{f^n(f(z))}{\lambda^n} = \frac{f^{n+1}(z)}{\lambda^n} = \lambda \varphi_{n+1}(z)$$

So if  $\varphi_n \to \varphi$ , then  $\varphi \circ f = \lambda \varphi$  and f(z) is conjugate to  $\lambda z$ . To see this convergence note that for |z| < r,

$$f(z) - \lambda z = a_2 z^2 + a_3 z^3 + \cdots$$
  
=  $z^2 (a_2 + a_3 z + a_4 z^2 + \cdots)$   
=  $z^2 g(z)$ 

where g(z) is analytic for |z|. Since g is continuous at 0, there exists  $\delta > 0$  such that,

$$\begin{aligned} |z| < \delta & \Rightarrow |g(z) - g(0)| < 1 \\ & \Rightarrow |g(z)| < 1 + |g(0)| \\ & \Rightarrow |g(z)| < 1 + |a_2| = C \end{aligned}$$

because  $g(0) = a_2$ . Now, for |z| < r,

$$|f(z) - \lambda z| = |z|^2 |g(z)| < C|z|^2$$

By making  $\delta$  smaller we can ensure  $C\delta < 1 - |\lambda|$ , so  $|\lambda| + C\delta < 1$  and also that  $\rho = \frac{(|\lambda| + C\delta)^2}{|\lambda|} < 1$ . For  $|z| < \delta$ 

$$\begin{aligned} |f(z)| &\leq |f(z) - \lambda z| + |\lambda z| \\ &\leq C|z|^2 + |\lambda| \cdot |z| \\ &= |z|(|\lambda| + C|z|) \\ &< \delta(|\lambda| + C|z|) \end{aligned}$$

where  $k = (|\lambda| + C|z|) < 1$ . From this we can see that,

$$|f(z)| < k\delta < \delta$$

for  $|z| < \delta$ . Therefore f(z) is in the domain of f. Also, f(f(z)) is defined and  $|f(f(z))| < k|f(z)| < k^2 \delta < \delta$ . By induction,

$$|f^n(z)| < k^n |z| < \delta$$

# 4.1. KOENING'S THEORENCHAPTER 4. LOCAL CONJUGATIONS AT FIXED POINTS Now,

$$\varphi_{n+1}(z) - \varphi_n(z) = \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} = \frac{f^{n+1}(z) - \lambda f^n(z)}{\lambda^{n+1}}$$

Now use  $|f(z) - \lambda z| \le C |z|^2$  and replace z with  $f^n(z)$  to get,

$$\begin{aligned} |\varphi_{n+1}(z) - \varphi_n(z)| &\leq \frac{f(f^n(z)) - \lambda f^n(z)}{|\lambda|^{n+1}} \\ &\leq \frac{C|f^n(z)^2|}{|\lambda|^{n+1}} \\ &\leq \frac{C(k^n|z|)^2}{|\lambda|^{n+1}} \\ &= \frac{k^{2n}C|z|^2}{|\lambda|^{n+1}} \\ &= \left(\frac{k^2}{|\lambda|}\right)^n \frac{C}{|\lambda|}|z|^2 \\ &= \rho^n \frac{C}{|\lambda|}|z|^2 \\ &\leq \frac{\rho^n C}{|\lambda|} \delta^2 \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} |\varphi_{n+1}(z) - \varphi_n(z)| \le \sum_{n=1}^{\infty} \frac{\rho^n C}{|\lambda|} \delta^2 < \infty$$

since  $\rho < 1$ . Thus  $\sum_{n=1}^{\infty} \varphi_{n+1}(z) - \varphi_n(z)$  converges uniformly on  $\{z \in \mathbb{C} : |z| < \delta\}$  by the Weierstrass M-test. Now,

$$\sum_{n=1}^{\infty} \varphi_{n+1}(z) - \varphi_n(z) = \lim_{n \to \infty} \sum_{n=1}^{N} \varphi_{n+1}(z) - \varphi_n(z)$$
$$= \lim_{N \to \infty} \left[ (\varphi_2(z) - \varphi_1(z)) + (\varphi_3(z) - \varphi_2(z)) + \dots + (\varphi_{N+1}(z) - \varphi_n(z)) \right]$$

which is a telescoping sum and therefore,

$$\sum_{n=1}^{\infty} \varphi_{n+1}(z) - \varphi_n(z) = \lim_{N \to \infty} (\varphi_{N+1}(z) - \varphi_1(z))$$

and so

$$\lim_{N\to\infty}\varphi_N(z)=\varphi(z)$$

## 4.1. KOENING'S THEOREMCHAPTER 4. LOCAL CONJUGATIONS AT FIXED POINTS

To show uniqueness, assume  $\varphi$  and  $\psi$  are two conjugations satisfying,

$$(\varphi \circ f \circ \varphi^{-1})(w) = \lambda w$$
 and  $(\psi \circ f \circ \psi^{-1})(z) = \lambda z$ 

We can say straight away from the left hand side that  $f(w) = \varphi^{-1}(\lambda \varphi(w))$ . From the right hand side we get  $f(\psi^{-1}(z)) = \psi^{-1}(\lambda z)$ . Now let  $w = \psi^{-1}(z)$ ,  $\psi(w) = z$  to get,

$$f(w) = \psi^{-1}(\lambda\psi(w))$$

Comparing with the above we deduce that,

$$\varphi^{-1}(\lambda\varphi(w)) = f(w) = \psi^{-1}(\lambda\psi(w))$$
  

$$\Rightarrow \varphi^{-1}(\lambda\varphi(w)) = \psi^{-1}(\lambda\psi(w))$$
  

$$\Rightarrow \varphi^{-1}[\lambda(\varphi \circ \psi^{-1})(z)] = \psi^{-1}(z)$$
  

$$\Rightarrow \lambda(\varphi \circ \psi^{-1})(z) = (\varphi \circ \psi^{-1})(\lambda z)$$
  

$$\Rightarrow \lambda\alpha(z) = \alpha(\lambda z)$$

where  $\alpha = \varphi \circ \psi^{-1}$  We know that  $\alpha$  is of the form  $\alpha(z) = 0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ , so

$$\lambda \alpha(z) = \lambda \alpha_1 z + \lambda \alpha_2 z^2 + \lambda \alpha_3 z^3 + \cdots$$
$$\alpha(\lambda z) = \lambda \alpha_1 z + \lambda^2 \alpha_2 z^2 + \lambda^3 \alpha_3 z^3 + \cdots$$

These are equal if for all  $n \ge 1$ 

$$\begin{aligned} \alpha_n \lambda^n &= \lambda \alpha_n &\Leftrightarrow \quad (\lambda^n - \lambda) \alpha_n \\ &\Leftrightarrow \quad (\lambda^n - \lambda) = 0 \text{ or } \alpha_n = 0 \end{aligned}$$

But  $\lambda^n \neq \lambda$  for  $n \ge 2$  since  $0 < \lambda < 1$  (specifically  $\lambda$  is not 0 or 1). So  $\alpha_n = 0$  for  $n \ge 2$ . This gives

$$\alpha_1 z = \alpha(z) = (\varphi \circ \psi^{-1})(z) = \varphi(\psi^{-1}(z))$$

Let  $\psi^{-1}(z) = w$  to get,

$$\varphi(z) = \alpha_1 \psi(z)$$

This completes the proof.

**Example 4.1.2.** Let  $a_1 = 0$  and  $a_2, \ldots, a_d \in \Delta$ . Suppose f(z) is a finite Blaschke product, i.e.

$$f(z) = e^{i\theta_0} z \prod_{j=2}^d \frac{z - a_j}{1 - \overline{a}_j z}$$

where each  $|a_j| < 1$ . For |a| < 1,  $\phi(z) = \frac{z-a}{1-\overline{a}z}$  is a Möbius transformation which maps the

4.1. KOENING'S THEOREMCHAPTER 4. LOCAL CONJUGATIONS AT FIXED POINTS unit circle to itself and maps  $\Delta \rightarrow \Delta$ . Therefore, for |z| < 1

$$|f(z)| = |z| \prod_{j=2}^d \left| \frac{z - a_j}{1 - \overline{a}_j z} \right| < 1$$

So,  $f : \Delta \to \Delta$ , f is analytic, and f(0) = 0. By the Schwarz lemma,  $|f'(0)| \le 1$  and if |f'(0)| = 1, then  $f(z) = \lambda z$  where  $|\lambda| < 1$ . The latter case only occurs when the product is empty. So, if  $d \ge 2$ , we have |f'(0)| < 1, so 0 is an attracting fixed point of f.

**Corollary 4.1.3.** If  $z_0$  is a repelling fixed point of f then (as with attracting points) there exists conformal  $\phi(z)$  mapping a neighbourhood of  $z_0$  onto a neighbourhood of 0 such that  $\phi$  conjugates f(z) to a linear function  $g(z) = \lambda z$ .

**Proof.** We know that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) = z_0 + \lambda(z - z_0)$$

from the linear approximation formula. Its inverse is given by,

$$f^{-1}(z) = z_0 + \frac{(z - z_0)}{\lambda}$$

because,

$$(f \circ f^{-1})(z) = f(f^{-1}(z)) = z_0 + \lambda \left( z_0 + \frac{(z - z_0)}{\lambda} - z_0 \right) = z_0 + z - z_0 = z$$

Because  $z_0$  is a repelling fixed point for f, i.e.  $|\lambda| > 1$ , it is therefore an attracting fixed point for  $f^{-1}$ . It is certainly a fixed point because,

$$f^{-1}(z_0) = z_0 + \frac{z_0 - z_0}{\lambda} = z_0$$

To prove it is attracting, note that,

$$(f^{-1}(z))' = \frac{1}{\lambda} \Rightarrow (f^{-1})'(z_0) = \frac{1}{\lambda} < 1$$

since  $|\lambda| > 1$ . We can thus apply Koenig's theorem to this map to say  $f^{-1}$  is conjugate to  $\frac{1}{\lambda}\phi$ . From this we get the following:

$$\begin{aligned} (\phi \circ f^{-1})(z) &= \frac{1}{\lambda}\phi(z) \\ \Rightarrow & \phi(z) = \frac{1}{\lambda}(\phi \circ f)(z) \\ \Rightarrow & \lambda\phi(z) = (\phi \circ f)(z) \end{aligned}$$

which means that f is conjugate to  $\lambda z$ .

### 4.1. KOENING'S THEOREMCHAPTER 4. LOCAL CONJUGATIONS AT FIXED POINTS

Now we have shown the existence of a nice conjugation for attracting and repelling fixed points. By this we mean that in a neighbourhood of the fixed point (be it repelling or attracting) the function f looks like a linear function, the behaviour of which, is easier to analyse.

**Example 4.1.4.** Any  $p(z) = z^m$  where  $m \ge 2$  has a repelling fixed point at z = 1 because  $p'(z) = mz^{m-1}$ , and  $|p'(1)| = m \ge 2$ . Then the conjugation function  $h(z) = \log z$  conjugates  $z^m$  to mz because,

$$(h \circ f)(z) = h(z^m) = \log z^m = m \log z = mh(z)$$

Its was shown in the early 1900s by Boettcher that a conjugation exists also for superattracting fixed points. Specifically a polynomial of degree  $d \ge 2$  which has a superattracting fixed point is conjugate to  $z^d$  in a neighbourhood of that point. Even more specifically, since  $\infty$  is a superattracting fixed point of any such p(z), it is therefore conjugate to  $z^d$  near  $\infty$ . The proof of this theorem is quite similar to the proof of Koenig's theorem and is available here [2, p 33].

## Chapter 5

## Newton Raphson Method

## 5.1 Introduction

Newton's approach to finding the roots of a polynomial can also be considered as a problem in dynamics. The iterates of the successively more accurate approximations were given by Newton as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where f is a polynomial. Finding the roots of f is the same a iterating a new function N(z) (defined below), and computing its orbits.

$$N(z) = z - \frac{f(z)}{f'(z)}$$

Note N is a rational function whenever f is a polynomial. By inspection, we can see that the zeroes of f correspond to fixed points of N. If  $z_0$  is a zero of f then  $f(z_0) = 0$  so  $N(z_0) = z_0$  and  $z_0$  is a fixed point. Conversely, if  $N(z_0) = z_0$  then  $\frac{f(z_0)}{f'(z_0)} = 0$  which implies  $f(z_0) = 0$  and z is a zero of f.

## 5.2 Quadratic polynomials

The simplest case to analyse is the case where f(z) is a quadratic. When f(z) has one repeated root, N(z) converges to that root regardless of how close the initial approximation is. However, when f(z) has two distinct roots, convergence is not automatic.

**Theorem 5.2.1.** If f(z) is a quadratic polynomial with one repeated root a, then N(z) converges to a for all initial points  $z \in \mathbb{C}$ .

**Proof.** Let  $p(z) = (z - a)^2$ , then p'(z) = 2(z - a) which means,

$$N(z) = z - \frac{(z-a)^2}{2(z-a)}$$
$$= \frac{z+a}{2}$$

## 5.2. QUADRATIC POLYNOMIALS CHAPTER 5. NEWTON RAPHSON METHOD

If we now iterate N(z) we get the sequence,

$$N(z) = \frac{z}{2} + \frac{a}{2}$$
$$N^{2}(z) = N(N(z)) = \frac{z}{4} + \frac{3a}{4}$$
$$N^{3}(z) = N(N(N(z))) = \frac{z}{8} + \frac{7a}{8}$$

The general formula for this sequence is thus given by,

$$N^{n}(z) = \frac{z}{2^{n}} + \frac{2^{n} - 1}{2^{n}}a$$

As  $n \to \infty$  the first term goes to 0 while  $\frac{2^n - 1}{2^n} \to 1$  which means that  $N^n(z) \to a$  for all  $z \in \mathbb{C}$ .

**Theorem 5.2.2.** If f(z) is a quadratic polynomial with distinct roots then f(z) is conjugate to the polynomial  $p(z) = z^2$ . [5, p 144]

**Proof.** Say f has roots a and b, then compose N with the Möbius transformation

$$h(z) = \frac{z-b}{z-a}$$

Here  $h(\infty) = 1$ , h(b) = 0 and  $h(a) = \infty$ . Then  $h \circ N \circ h^{-1}$  is a rational map of degree 2 that has fixed points at 1, 0 and  $\infty$ , with the latter two being superattracting. It is therefore the map  $p(z) = z^2$ . So  $h \circ N \circ h^{-1}(z) = p(z)$  and N is conjugate to p.

Under this conjugation,  $a \to \infty$  and  $b \to 0$ . It also takes the perpendicular bisector of the line from a to b, to the unit circle. By comparing with the  $z^2$  example, we can state that the basin of attraction of a is the half-plane on its side of the bisector. Similarly the basin of attraction for b is the half-plane on its side of the bisector.

#### Table 5.1: Conjugacy of Newton's method for quadratics. [5, p 144]

It is also evident that if our initial approximation to the root isn't in either of these half planes it must be on the bisector, which isn't in either basin of attraction and will therefore not converge to either of the roots. Newton's method will fail if the initial approximation is on this line, i.e. if the approximation is equidistant from a and b.

**Example 5.2.3.** Take  $f(z) = (z - 1)(z - 3) = z^2 - 4z + 3$  which has roots at the points 1 and

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3. Then f'(z) = 2z - 4, and

$$N(z) = z - \left(\frac{z^2 - 4z + 4}{2z - 4}\right) = \frac{z^2 - 3}{2z - 4}$$

where N(1) = 1 and N(3) = 3. If we start at the point 0 say which is in the half-plane containing the root at 1 (and also inside the unit disk under the conjugation discussed before), the iterates of N should get closer and closer to 1. Evidently,

$$N(0) = \frac{-3}{-4} = \frac{3}{4}$$
$$N^{2}(0) = N(N(0)) = N\left(\frac{3}{4}\right) = \frac{\left(\frac{3}{4}\right)^{2} - 3}{2\left(\frac{3}{4}\right) - 4} = \frac{39}{40}$$
$$N^{3}(0) = N(N^{2}(0)) = N\left(\frac{39}{40}\right) = \frac{\left(\frac{39}{40}\right)^{2} - 3}{2\left(\frac{39}{40}\right) - 4} = \frac{3279}{3280}$$

which does indeed approach 1 as  $n \to \infty$ . The perpendicular bisector of the line between the two roots is the line  $\{z \in \mathbb{C} : \Re(z) = 2\}$  whose points are all equidistant from 1 and 3. If we start with an initial approximation on this line, then N(z) should fail to converge to a root of f. Take z = 2. Then,

$$N(2) = \frac{4-3}{4-4} = \infty$$

and the iterates don't converge to either root as expected.

## 5.3 Cubic Equation

Analysing Newton's method quickly becomes far more complicated as the degree of the polynomial in question gets larger. We'll motivate the study of the case where p is a cubic equation by the following example.

**Example 5.3.1.** We'll first take an example of a cubic polynomial which has 3 simple roots (roots of multiplicity 1). Let

$$p(z) = (z-1)(z-2)(z-3) = z^3 - 6z^2 + 11z - 6$$

which has three roots at the points 1, 2 and 3. Then,

$$p'(z) = 3z^2 - 12z + 11$$

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which means the function N(z) is as follows:

$$N(z) = z - \frac{p(z)}{p'(z)}$$
$$= z - \frac{z^3 - 6z^2 + 11z - 6}{3z^2 - 12z + 11}$$
$$= \frac{2z^3 - 6z^2 + 6}{3z^2 - 12z + 11}$$

The claim is that this function has superattracting fixed points at 1, 2 and 3. We'll check they are fixed points.

$$N(1) = \frac{2-6+6}{3-12+11} = 1$$
$$N(2) = \frac{16-24+6}{12-24+11} = \frac{-2}{-1} = 2$$
$$N(3) = \frac{54-54+6}{27-36+11} = \frac{6}{2} = 3$$

With a relatively small amount of work (but a lengthy amount of calculus) one can compute N'(z) at these points and see that they are indeed superattracting. Rather than waste time with this calculation, we'll prove the following theorem for this purpose.

**Theorem 5.3.2.** Any simple zero of p is a superattracting fixed point of N.

**Proof.** We've seen that the zeroes of p are fixed points of N. To show they are superattracting, we compute as follows:

$$N'(z) = \left(z - \frac{p(z)}{p'(z)}\right)'$$
  
=  $1 - \left(\frac{p'(z)p'(z) - p(z)p''(z)}{p'(z)^2}\right)$   
=  $\frac{p'(z)^2 - p'(z)^2 + p(z)p''(z)}{p'(z)^2}$   
=  $\frac{p(z)p''(z)}{p'(z)^2}$ 

Which is itself equal to 0 at the simple zeroes of p. (i.e. when p(z) = 0 and  $p'(z) \neq 0$ ).

**Theorem 5.3.3.** Any zero of p is an attracting fixed point of N. [3, p 23]

**Proof.** Assume that  $p(z) = (z - \zeta)^m g(z)$  where  $m \ge 1$ ,  $g(\zeta) \ne 0$ , and  $\zeta$  is a root of p with

multiplicity m. We'll calculate N'(z) at the point  $\zeta$ .

$$p'(z) = m(z - \zeta)^{m-1}g(z) + (z - \zeta)^m g'(z)$$

$$p''(z) = m(m-1)(z - \zeta)^{m-2}g(z) + 2m(z - \zeta)^{m-1}g'(z) + (z - \zeta)^m g''(z)$$

$$p(z)p''(z) = m(m-1)(z - \zeta)^{2m-2}g(z)^2 + (z - \zeta)^{2m}g(z)g''(z) + 2m(z - \zeta)^{2m-1}g'(z)g(z)$$

$$p'(z)^2 = m^2(z - \zeta)^{2m-2}g(z)^2 + (z - \zeta)^{2m}g'(z) + 2m(z - \zeta)^{2m-1}g(z)g'(z)$$

So,

$$N'(z) = \frac{m(m-1)(z-\zeta)^{2m-2}g(z)^2 + (z-\zeta)^{2m}g(z)g''(z) + 2m(z-\zeta)^{2m-1}g'(z)g(z)}{m^2(z-\zeta)^{2m-2}g(z)^2 + (z-\zeta)^{2m}g'(z) + 2m(z-\zeta)^{2m-1}g(z)g'(z)}$$
  
$$\Rightarrow N'(\zeta) = \frac{m(m-1)}{m^2} = \frac{m-1}{m}$$

Hence,  $0 \leq N'(\zeta) < 1$  for all  $m \geq 1$ . Thus,  $\zeta$  is an attracting fixed point of p.

The case m = 1 (i.e. when  $\zeta$  is a simple zero) in the above proof means  $N'(\zeta) = 0$  and thus  $\zeta$  is superattracting which agrees with the previous theorem.

What we've seen here is the following: In Newton's method, if the initial approximation to a root is chosen within the basin of attraction then we have shown that the method will converge to the attracting fixed point of N(z), which is the root. However, if the initial point is outside of any basin of attraction then Newton's method will fail to converge.

As a final example take the cubic equation  $f(z) = z^3 - 1$ . This has three roots namely, 1,  $\frac{1\pm\sqrt{3}i}{2}$ . However, unlike the quadratic case, the basins of attraction for these points are somewhat complicated. The Julia set here is no longer a simple curve.

Table 5.2: Julia set for  $f(z) = z^3 - 1$ . [5, p 140]

The white line in the above diagram is the Julia set for N. As we can see, if we start at 0, we won't converge to any of the roots (i.e. the fixed points). Indeed,

$$p(z) = z^3 - 1 \qquad p'(z) = 3z^2$$
  

$$\Rightarrow \qquad N(z) = \frac{3z^2 + 1}{3z^3}$$
  

$$\Rightarrow \qquad N(0) = \infty$$

This brings to an end our discussion of Newton's method. Viewing the method as a dynamical

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system is a very sleek way of describing why it might not work for certain choices of the initial approximation to the root. In particular, the simplicity of the quadratic case when we conjugate to  $p(z) = z^2$  I found somewhat surprising.

## Chapter 6

# Structure of the Julia Set

## 6.1 Invariance

**Definition 6.1.1** (Invariance). If f is map from a set X to itself, a subset U is called:

- forward invariant if f(U) = U
- backward invariant if  $f^{-1}(U) = U$
- (completely) invariant if it is both forward and backward invariant.

If f is surjective, then backwards and complete invariance are synonymous. Suppose f is surjective. Then

$$f(f^{-1}(U)) = U$$

If f is also backward invariant, then  $f^{-1}(U) = U$ , which implies

f(U) = U

Therefore, f is completely invariant. [4, Ch 8]

**Theorem 6.1.2.** The Fatou set F(R) of a rational map R is completely invariant, as is its complement the Julia set.

**Proof.** We will show that F(R) is invariant first. This implies that its complement J(R) is also invariant. Since R is surjective we need only show backwards invariance. We'll show that,  $R^{-1}(F(R)) = F(R)$  by showing that both sets are included in each other and therefore equal. Take  $z_0 \in R^{-1}(F(R))$ . Now let  $w_0 = R(z_0)$ , which is in F(R). Therefore, given  $\epsilon > 0$ , there exists  $\delta > 0$  such that,

$$\sigma(w, w_0) < \delta \qquad \Rightarrow \qquad \sigma(R^n(w), R^n(w_0)) < \epsilon$$

$$\begin{aligned} \sigma(z, z_0) < \eta &\Rightarrow & \sigma(R(z), R(z_0)) < \delta \\ &\Rightarrow & \sigma(R(z), w_0) < \delta \\ &\Rightarrow & \sigma(R^{n+1}(z), R^n(w_0)) < \epsilon \\ &\Rightarrow & \sigma(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon \end{aligned}$$

for n = 1, 2, ... So  $\{R^{n+1} : n \ge 1\}$  is equicontinuous at  $z_0$ . Thus,  $\{R^n : n \ge 1\}$  is equicontinuous at  $z_0$  because R is continuous at  $z_0$  by above. Hence,  $\{R^n : n \ge 1\}$  is equicontinuous on  $R^{-1}(F(R))$ , which gives that

$$R^{-1}(F(R)) \subseteq F(R)$$

To prove the reverse inclusion, let  $w_0 \in R(F(R))$ . Then there exists  $z_0 \in F$  such that  $w_0 = R(z_0)$ . Because  $z_0 \in F(R)$ , given  $\epsilon > 0$ , there exists  $\delta > 0$  such that,

$$\sigma(z, z_0) < \delta \qquad \Rightarrow \qquad \sigma(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$$

The set  $U = \{z \in \hat{\mathbb{C}} : \sigma(z, z_0) < \delta\}$  is an open neighbourhood of  $z_0$ , so R(U) is an open neighbourhood of  $R(z_0) = w_0$  by the Open Mapping Theorem. If  $w \in R(U)$ , then w = R(z) for some  $z \in U$ , so,

$$\sigma(R^{n}(w), R^{n}(w_{0})) = \sigma(R^{n+1}(z), R^{n+1}(z_{0})) < \epsilon$$

which implies  $w \in F(R)$  and therefore  $F(R) \subseteq R^{-1}(F(R))$ . Thus,  $F(R) = R^{-1}(F(R))$  and F(R) is backwards invariant, and hence completely invariant.

## 6.2 The Exceptional Set

This section is based on [4, Ch 8].

**Definition 6.2.1** (Grand Orbit). The grand orbit [z] of a point z is given by:

$$[z] = \{ w \in \hat{\mathbb{C}} : O^+(z) \cap O^+(w) \neq \emptyset \}$$

Given a point  $z \in \hat{\mathbb{C}}$ , this is the set of other points whose orbits eventually coincide with the orbit of z. In particular all the inverse iterates of z

$$O^{-}(z) = \{ \zeta \in \hat{\mathbb{C}} : z \in O^{+}(\zeta) \}$$

are in [z] since their forward orbits will eventually coincide with that of z by definition.

**Definition 6.2.2** (Exceptional Point). If [z] is finite then, z is called an *exceptional point*. The set of such points is called the exceptional set and is denoted E(R).

If [z] is finite then  $O^{-}(z) \subseteq [z]$  must be finite since it is the subset of a finite set. We'll

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see shortly that the connection between [z] and  $O^{-}(z)$  is important for what we want to show. We'll focus mainly on the case where R is a polynomial for our discussion of the Julia set. The reason for this will become clear. We'll prove several details of the exceptional set in quick succession now and use them throughout the rest of the section.

**Proposition 6.2.3.** For a polynomial p, if  $p^{-1}(E(p)) \subseteq E(p)$  and  $p(E(p)) \subseteq E(p)$ , then E(p) is completely invariant.

**Proof.** To see this, note that since polynomials are surjective maps from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}$ , for some  $w \in \hat{\mathbb{C}}$  we have

$$z \in E(p) \subseteq \mathbb{C} \quad \Rightarrow \quad z = p(w)$$
  
$$\Rightarrow \quad w \in p^{-1}(E(p)) \subseteq E(p)$$
  
$$\Rightarrow \quad z \in p(E(p))$$

So  $E(p) \subseteq p(E(p))$  and  $p(E(p)) \subseteq E(p)$  gives p(E(p)) = E(p). It follows that  $E(p) \subseteq p^{-1}(E(p))$ and since  $p^{-1}(E(p)) \subseteq E(p)$ , we get  $p^{-1}(E(p)) = E(p)$  and E(p) is invariant.

**Proposition 6.2.4.** For a polynomial p with deg  $(p) \ge 1$ , and any  $z \in \hat{\mathbb{C}}$ , the grand orbit [z] is a completely invariant set.

**Proof.** If  $w \in [z]$ , then  $p^n(w) = p^m(z)$  for some integers m, n > 0. This implies  $p(w) \in [z]$  since,

$$p^{n}(p(w)) = p(p^{n}(w)) = p(p^{m}(z)) = p^{m+1}(z)$$

Therefore, we have  $p([z]) \subseteq [z]$ . Now suppose  $u \in p^{-1}([z])$  and p(u) = w for some  $w \in [z]$ . Then we have,

$$p^{n+1}(u) = p^n(w) = p^m(z)$$

Therefore  $u \in [z]$ , and hence  $p^{-1}([z]) \subseteq [z]$ 

**Proposition 6.2.5.** [z] is the smallest invariant set containing z.

**Proof.** Let E be any invariant set containing z. Then E contains all the forward and reverse iterates of z, in particular  $p^n(z) \in E$  for all n. Obviously  $z \in [z]$ . If w is some other point in [z] then  $p^n(z) = p^m(w)$  for some  $n, m \ge 0$  (Note both n > m, n = m and n < m are possible and these are assumed to be the lowest choice of n and m for which this holds). If m = 0 we immediately have  $w = p^n(z) \in E$ . If  $m \ge 1$  we have,

$$p^{m}(w) = p^{n}(z)$$

$$\Rightarrow p^{-1}(p^{m}(w)) = p^{-1}(p^{n}(z))$$

$$\Rightarrow p^{m-1}(w) = p^{-1}(p^{n}(z))$$

$$\Rightarrow p^{m-1}(w) \in p^{-1}(E) \subseteq E$$

The first implication might not seem obvious until one realises that in this case (where  $m \neq 0$ ), taking  $p^{-1}(p^n(z))$  gives two points, one in the orbit of z and one in the orbit of w. These points

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are distinct but the above is true for one of them. By induction on m we have  $w \in E$ , and so  $[z] \subseteq E$  and so any invariant set containing z must also contain [z].

**Theorem 6.2.6.** For a polynomial p, with deg  $(p) \ge 2$ , E(p) has at most two points.

**Proof.**  $\infty$  is an exceptional point because  $[\infty] = \{\infty\}$ . If there exists a finite point  $z \in E(p)$ , then p must act as a permutation of the finite set [z]. Hence some iterate  $p^n$  will be the identity on [z]. By complete invariance of [z] the only solution w of  $p^n(w) = z$  is w = z. Suppose  $p^n$  has degree d. Then  $p^n(w) = z$  has d solutions counting multiplicities. Hence,

$$p^n(w) - z = a(w - z)^d$$

for some  $a \neq 0$ , and there is at most one z with this property. Therefore, there is at most one finite point in E(p).

**Example 6.2.7.** We've seen  $p(z) = z^2$  a few times already. The grand orbit of the fixed point 0 is actually just  $\{0\}$  itself. This is because p(0) = 0 and p(z) = 0 implies that z = 0. The orbit of 0 is the singleton set  $\{0\}$  because it is a fixed point. Despite the fact that 0 is a superattracting fixed point (and thus points in the basin of attraction of 0 converge to it as  $n \to \infty$ ), there are no other points whose orbits actually contain 0. Thus the exceptional set is given by  $E(p) = \{0, \infty\}$ 

**Example 6.2.8.** Take the polynomial  $p(z) = 2 + 4(z-2)^2$ . The grand orbit of the fixed point at 2 is finite because p(2) = 2 and

$$p(z) = 2 \quad \Leftrightarrow \quad 4(z-2)^2 = 0 \quad \Leftrightarrow \quad z = 2$$

So  $[2] = \{2\}$  and  $E(p) = \{2, \infty\}$ .

We already know a good amount about the exceptional set from the conclusions just made. However, it is of utmost importance to understand how this set fits in with the already established Julia and Fatou sets if we are to use it to our advantage. Given that the union of the Julia and Fatou set of a polynomial is the whole of  $\hat{\mathbb{C}}$ , where does the exceptional set fit in?

**Theorem 6.2.9.** If p is a polynomial with deg  $(p) \ge 2$  then  $E(p) \subseteq F(p)$ , i.e. all the exceptional points lie in the Fatou set.

**Proof.** We have proved that  $\infty \in F(p)$ . From the previous theorem, if there is a finite point in E(p) it must be a superattracting fixed point of some iterate  $p^n$ . Therefore it must be in  $F(p^n)$ . We claim that  $F(p^n) = F(p)$ . Firstly it is fairly obvious that  $F(p) \subseteq F(p^n)$  since the iterates of  $p^n$  are a subset of the iterates of p. To see this note that,

$$\underbrace{p^n \circ p^n \circ \dots \circ p^n}_{k \text{ times}} = p^{kn}$$

# 6.2. THE EXCEPTIONAL SET CHAPTER 6. STRUCTURE OF THE JULIA SET So, F(p) is the largest open set on which the family of iterates $\{p, p^2, p^3, ...\}$ is equicontinuous, whereas $F(p^n)$ is the largest open set on which the family of iterates $\{p^n, p^{2n}, p^{3n}, ...\}$ is equicontinuous. Since $n \ge 1$ , the second set of iterates is contained in the first, i.e.

$$\{p^n, p^{2n}, p^{3n}, \dots\} \subseteq \{p, p^2, p^3, \dots\}$$

Since there are extra functions in the bigger set, it is harder for that family (the family of iterates of p) to be equicontinuous and thus the Fatou set for p can't be larger than the Fatou set for the smaller family of iterates of  $p^n$ . Therefore,  $F(p) \subseteq F(p^n)$ . To see the reverse inclusion first assume the iterates  $\{p^{nk} : k \ge 0\}$  are equicontinuous on  $F(p^n)$ . Then given any  $\eta_0 > 0$ , there exists  $\delta_0 > 0$  such that,

$$\sigma(z, z_0) < \eta_0 \quad \Rightarrow \quad \sigma(p^{nk}(z), p^{nk}(z_0)) < \delta_0$$

Note that  $\eta_0, \delta_0$  are chosen instead of the usual  $\delta, \epsilon$  in order to simplify forthcoming notation. Now,  $p^j$  is uniformly continuous on  $\hat{\mathbb{C}}$  means given any  $\epsilon$  there exists  $\delta_j > 0$  such that,

$$\sigma(w_1, w_2) < \delta_j \quad \Rightarrow \quad \sigma(p^j(w_1), p^j(w_2)) < \epsilon$$

We know from equicontinuity of  $\{p^{nk} : k \ge 0\}$  that given  $\eta_j > 0$ , there exists  $\delta_j >$  such that,

$$\sigma(z, z_0) < \eta_j \quad \Rightarrow \quad \sigma(p^{nk}(z), p^{nk}(z_0)) < \delta_j$$

and from uniform continuity,

$$\sigma(z, z_0) < \eta_j \quad \Rightarrow \quad \sigma(p^j(p^{nk}(z)), p^j(p^{nk}(z_0))) < \epsilon_j$$

Therefore  $p^{nk+j}$  is equicontinuous on the same set as  $\{p^{nk} : k \ge 0\}$  where  $k \ge 0$  and j = 0, 1, ..., n-1. Because a finite union of equicontinuous families is equicontinuous, it follows that  $\{p^m : m \ge 0\}$  is equicontinuous on the same set and thus  $F(p^n) \subseteq F(p)$ . Hence,  $F(p^n) = F(p)$ .

**Proposition 6.2.10.** If p a polynomial of degree at least one and  $z \in \hat{\mathbb{C}}$ , then  $O^{-}(z)$  is finite iff  $z \in E(p)$ .

**Proof.** This is one of the most important theorems in this section. We've already stated that for  $z \in E(p)$ ,  $O^{-}(z)$  is finite. For the converse, assume  $O^{-}(z)$  is finite, and let,

$$B_n = \bigcup_{m \ge n} p^{-m} \{z\} = \{ w \in \hat{\mathbb{C}} : p^m(w) = z, m \ge n \}$$

This is the set of backwards iterates of the point z which go backwards at least n iterations (i.e. they are at least n iterations away from z.) If n is large we have fewer inverse iterates, but as n gets lower we include inverse iterates which are closer and closer to z (in terms of iteration),

6.2. THE EXCEPTIONAL SET CHAPTER 6. STRUCTURE OF THE JULIA SET thus,

$$O^{-}(z) = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$$

Because  $O^{-}(z)$  is finite we must have  $B_{n+1} = B_n$  for some n, otherwise we would be taking strict subsets of a finite set an infinite number of times which isn't possible. Saying  $B_{n+1} = B_n$ is equivalent to saying that the inverse iterates only go back so far, after which point there are no more (in other words that  $O^{-}(z)$  is finite). From this we get  $p^{-1}(B_n) = B_n$  which means pmust permute the elements of  $B_n$  which is thus invariant.

Because of this invariance we can say that  $z \in p^m(B_n) \subseteq B_n$ , and thus  $[z] \in B_n$  since any invariant set containing z must contain [z] by proposition 6.2.5.

## Chapter 7

# **Normal Families**

## 7.1 Preliminary Work

**Proposition 7.1.1.** If D is a union of components of F(R), (where a component is a maximal, connected subset of F(R)) such that D is invariant, then  $\partial D = J(R)$ .

**Proof.** Any completely invariant subset of J(R) is dense in J(R). Because  $\partial D$  is an invariant subset of J(R), it must also be dense in J(R). But  $\partial D$  is closed so  $\partial D = J(R)$ .

This proof is adapted from [2, p 57]. We can prove a more concise version of this proposition by using Montel's theory of normal families.

## 7.2 Normal Families

The theory of normal families was initially formalised by Montel but put to extensive use by Fatou and Julia in the study of complex dynamics. The Fatou and Julia sets can be defined using normal families rather than equicontinuous families. It is unsurprising therefore that there is a close relationship between the two concepts. However, the more powerful assumption of normality is important for some subsequent results and discussion.

**Definition 7.2.1** (Normal). A family of functions  $\{F_i : i \in I\}$  from a metric space  $(X, d_X)$  to another metric space  $(Y, d_Y)$ , is said to be *normal* in X if every infinite sequence of functions from  $\{F_i : i \in I\}$  contains a subsequence which converges locally and uniformly (on every compact subset) on X.

To emphasise the connection between normal families and equicontinuous ones, we'll state the following theorem.

**Theorem 7.2.2** (Arzela-Ascoli). Let  $G \subseteq \hat{\mathbb{C}}$  be a domain, i.e. an open, connected set. A family  $\mathcal{F}$  of continuous functions with values in  $(\hat{\mathbb{C}}, \sigma)$  is normal on G iff  $\mathcal{F}$  is equicontinuous on every compact set  $K \subseteq G$ .

This is a version of the standard Arzela-Ascoli theorem in Complex Analysis which is simplified greatly here because we're working in the Riemann sphere which is compact. Already

#### 7.2. NORMAL FAMILIES

we're in a position to prove a quite useful theorem this time due to another stalwart of both Complex Analysis and Dynamics, Paul Montel. We'll do this a little later on.

**Definition 7.2.3** (Connectedness). A set X is called *connected* if there does not exist proper closed subsets  $A, B \subseteq X$  such that  $A \cup B = X$  but  $A \cap B = \emptyset$ .

**Definition 7.2.4** (Simple Connectedness). A set X is called *simply connected* if it is connected and every closed curve can be continuously deformed into some constant curve in X.

**Proposition 7.2.5.** Let  $G \subseteq \hat{\mathbb{C}}$  be an open subset of the Riemann sphere. Then  $\hat{\mathbb{C}} \setminus G = G^c$  is connected iff each component of G is simply connected.

This will be particularly useful if we let G be the Fatou set F(R) of some rational function R. Then the proposition becomes the following: J(R) is connected iff each component of F(R) is simply connected [3, p 81].

**Definition 7.2.6** (*d*-fold Map). A *d*-fold map f of V onto W is a map where for all  $w \in W$ , f(z) = w has d solutions in V (counting multiplicities). For example any rational function R(z) of deg d is a d-fold map.

**Theorem 7.2.7.** If U is a completely invariant component of F(R), then  $\partial U = J(R)$ . Furthermore, every other component of F(R) is simply connected and there are at most two completely invariant subsets of F(R).

Before we prove this theorem we'll need to familiarise ourselves with the Riemann-Hurwitz relation. This says that a rational map of degree d has at most 2d - 2 critical points. If we compose R with a Möbius transformation  $\phi$  (which doesn't change the critical points) so that  $\tilde{R}(z) = (\phi \circ R)(z) = \frac{P(z)}{Q(z)}$  we can reduce to the case where  $\tilde{R}(\infty) = 0$ . This implies that deg (P) is strictly less than deg (Q), and furthermore that  $\tilde{R}$  is of the following form,

$$\tilde{R}(z) = \frac{P(z)}{Q(z)} = \frac{\alpha z^{d-1} + \cdots}{\beta z^d + \cdots}$$

Then we have,

$$\tilde{R}'(z) = \frac{Q(z)P'(z) - P(z)Q'(z)}{Q(z)^2} = \frac{-\alpha\beta z^{2d-2} + \cdots}{Q(z)^2}$$

And so  $\tilde{R}$  (and hence R) has 2d - 2 critical points (counting multiplicities) [2, p 54]. **Proof.** [of 7.2.7] The first part is true from the proposition 7.1.1. If we consider  $\{R^n : n > 0\}$ on  $\hat{\mathbb{C}} \setminus \overline{U}$ , then  $\{R^n : n > 0\}$  is normal there because

$$J(R) = \partial U \not\subseteq \hat{\mathbb{C}} \setminus \overline{U}$$

Also  $\hat{\mathbb{C}} \setminus \overline{U} \subseteq F(R)$  since  $\{R^n : n > 0\}$  is also normal on U by assumption but U isn't in  $\hat{\mathbb{C}} \setminus \overline{U}$ . Since U is connected, each component of  $\hat{\mathbb{C}} \setminus \overline{U}$  must be simply connected. If we consider U to be a simply connected invariant component then R is a d-fold map of U onto U. Thus U must contain d-1 critical points and since there are at most 2d-2 critical points, there are at most two such components. [2, p 70]

Conveniently, we are now in a position to prove an assumption from our discussion of the Mandelbrot set in a previous chapter. We stated that for a polynomial of the form  $p(z) = z^2 + c$ , if the orbit of 0 stays bounded then J(p) is connected. Equivalently, we could say that if 0 isn't in the basin of attraction for  $\infty$ , then J(p) is connected. This is because 0 is the only finite critical point of this map.

**Theorem 7.2.8.** For p(z) with degree  $p \ge 2$ , if the basin of attraction of  $\infty$  contains no finite critical points then J(p) is connected.

**Proof.** Denote by  $A(\infty)$ , the basin of attraction of the superattracting fixed point at  $\infty$ . Now we'll use theorem 7.2.7. We know from Boettcher's argument in example 4.1.4 that p(z) is conjugate to  $\zeta^d$  in a neighbourhood of  $\infty$  under some conjugation function  $\varphi$ . If there are no finite critical points in  $A(\infty)$ , then we can extend  $\varphi$  to the whole of  $A(\infty)$ . Then  $\varphi: A(\infty) \to \{\zeta \in \hat{\mathbb{C}} : |\zeta| > 1\}$  which is the complement of the closed unit disk. This implies that  $A(\infty)$  is simply connected. Thus  $\partial A(\infty) = J(p)$  is connected [2, p 65,66].

## 7.3 Montel's Theorem

For the next section we'll first state and prove a variant of Montel's theorem and then extend it to the Riemann sphere. The proof usually relies on the more general version of the Arzela-Ascoli theorem stated earlier, but our version will not. The first theorem below was taken from [4, Ch 5], and the subsequent theory is adapted from [4, Ch 8].

**Theorem 7.3.1** (Montel for  $\mathbb{C}$ ). Let  $G \subseteq \mathbb{C}$  be open and  $\mathcal{F}$  a family of analytic functions

$$\mathcal{F} = \{ f : G \to G \mid f \text{ is analytic} \}$$

If there exists finite M with  $\sup_{z \in G} |f(z)| < M$  for all  $f \in \mathcal{F}$  then  $\mathcal{F}$  is equicontinuous on G.

**Proof.** For  $z_0 \in G$  there exists r > 0 such that  $B(z_0, r) \subseteq G$ . Choose  $\delta_0 < \frac{r}{2}$ , so  $2\delta_0 < r$ . Then we have,

$$\overline{B}(z_0, 2\delta_0) \subseteq B(z_0, r) \subseteq G$$

From the Cauchy integral formula we have,

$$f'(z) = \frac{1}{2\pi} \int_{|\zeta - z_0| = 2\delta_0} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

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for  $z \in B(z_0, 2\delta_0)$ . If we take  $z \in B(z_0, \delta_0)$ , then,

$$\begin{aligned} f'(z)| &= \left| \frac{1}{2\pi} \int_{|\zeta - z_0| = 2\delta_0} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \\ &\leq \frac{1}{2\pi} (2\pi) (2\delta_0) \sup_{|\zeta - z| = 2\delta_0} \frac{|f(\zeta)|}{(\zeta - z)^2} \\ &\leq 2\delta_0 \frac{M}{\delta_0^2} \\ &= \frac{2M}{\delta_0} \end{aligned}$$

So for  $|z - z_0| < \delta_0$  we have,

$$|f(z) - f(z_0)| = \left| \int_{z_0}^z f'(\zeta) \, d\zeta \right|$$
  
$$\leq |z - z_0| \sup_{\substack{|\zeta - z_0| = 2\delta_0 \\ |\zeta - z_0| = 2\delta_0}} |f'(\zeta)|$$
  
$$\leq |z - z_0| \frac{2M}{\delta_0}$$

So given  $\epsilon > 0$ , if we choose  $\delta = \min\left\{\delta_0, \epsilon \frac{\delta_0}{2M}\right\}$  we get,

$$|z - z_0| < \delta_0 \qquad \Rightarrow \qquad |f(z) - f(z_0)| < \epsilon$$

for all  $f \in \mathcal{F}$ . Thus  $\mathcal{F}$  is equicontinuous on G.

We'll now use Montel's theorem to prove and equivalent version for the Riemann sphere. As might be expected, the only real obstacle to deal with, is the treatment of the extra point at  $\infty$ .

**Theorem 7.3.2** (Montel for  $\hat{\mathbb{C}}$ ). If  $G \subseteq \hat{\mathbb{C}}$  and  $H \subseteq \hat{\mathbb{C}}$  are open and not the empty set then,

$$\mathcal{F} = \{ f : G \to \hat{\mathbb{C}} \mid f \text{ is holomorphic and } f(G) \cap H = \emptyset \}$$

is equicontinuous on G.

**Proof.** If  $\infty \in H$  then there exists R > 0 with

$$\{z \in \mathbb{C} : |z| > R\} \subseteq (H \cap \mathbb{C}) = H \setminus \{\infty\}$$

Then  $f(G) \subseteq \overline{B}(0, R)$  for all  $f \in \mathcal{F}$ . Then by the standard version of Montel's theorem,  $\mathcal{F}$  is equicontinuous at each point of G with respect to the Euclidean metric, i.e. given  $z_0 \in G$  and

 $\epsilon > 0$  there exists  $\delta > 0$  such that.

$$\begin{aligned} |z - z_0| < \delta & \Rightarrow & |f(z) - f(z_0)| < \epsilon \\ & \Rightarrow & \sigma(f(z), f(z_0)) < \epsilon \end{aligned}$$

since  $\sigma(f(z), f(z_0)) < 2|f(z) - f(z_0)|$ . Thus  $\mathcal{F}$  is also equicontinuous with respect to the chordal metric. If  $\infty \notin H$ , choose  $w_0 \in H$  and let  $\mathcal{E}$  be the family,

$$\mathcal{E} = \left\{ g(z) = \frac{1}{f(z) - w_0} : f \in \mathcal{F} \right\}$$

Then we have a set  $\tilde{H} = \left\{ \frac{1}{w - w_0} : w \in H \right\}$  such that  $g(G) \cap \tilde{H} = \emptyset$  and  $\infty \in \tilde{H}$ . Thus  $\mathcal{E}$  is equicontinuous on G by the first part. Now define  $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$  by  $\phi(\zeta) = \frac{1}{\zeta} + w_0$ . Then  $\phi \circ g = f$  since,

$$\phi(g(z)) = \phi\left(\frac{1}{f(z) - w_0}\right) = f(z) - w_0 + w_0 = f(z)$$

and  $\phi$  is uniformly continuous on  $\hat{\mathbb{C}}$ . By uniform continuity, given  $\epsilon > 0$ , there exists  $\delta > 0$  such that,

$$\sigma(g(z),g(z_0)) < \delta \qquad \Rightarrow \qquad \sigma(\phi(g(z)),\phi(g(z_0)) < \epsilon$$

for all  $g \in \mathcal{E}$ . Hence,  $\mathcal{F} = \{\phi \circ g : g \in \mathcal{E}\}$  is equicontinuous on G.

**Theorem 7.3.3.** If p is a polynomial with deg  $(p) \ge 2$ , then J(p) has empty interior.

We've seen two examples of such polynomials already  $(p(z) = z^2 \text{ and } p(z) = z^2 - 2)$  whose Julia set is a simple curve and thus has empty interior.

**Proof.** Let  $G = (J(p))^{\circ} \subseteq J(p)$ . Since J(p) is invariant, the values of the iterates  $\{p^n : n \ge 1\}$  restricted to G can't intersect F(p) so  $p^n(G) \cap F(p) = \emptyset$  for all  $n \ge 1$ . Because F(p) is open and non-empty for deg  $(p) \ge 2$  (since  $\infty \in F(p)$ ), we can apply Montel's theorem with H = F(p) to say that the family,

$$\mathcal{F} = \{p^n : n \ge 1\}$$

is equicontinuous on G. This is due to the fact that polynomials are holomorphic on  $\mathbb{C}$  and  $p^n(G) \cap F(p) = \emptyset$  as stated already. Thus  $G \subseteq F(p)$  and  $G \subseteq J(p)$  which is a contradiction. Hence  $G = \emptyset$  and J(p) has empty interior.

### 7.4 Inverse Iteration Method

**Theorem 7.4.1** (Montel/Picard). If  $G \subseteq \hat{\mathbb{C}}$  is open and  $a, b, c \in \hat{\mathbb{C}}$  are three distinct points then,

$$\mathcal{F} = \{ f : G \to \hat{\mathbb{C}} \mid f \text{ is holomorphic and } f(G) \cap \{a, b, c\} = \emptyset \}$$

is equicontinuous on G.

## 7.4. INVERSE ITERATION METHOD

Proof omitted, see [1, p 76].

This bears a resemblance to Montel's theorem which we proved in the previous section. However, the specific choice of three points  $\{a, b, c\}$  rather than an unknown set makes this theorem highly useful. We omit the proof as it is quite involved and would constitute a significant digression from the topics covered here. We shall however, make extensive use of this theorem to prove some subsequent ones.

**Corollary 7.4.2.** If p is a polynomial with deg  $(p) \ge 2$ ,  $z \in J(p)$  and U is an open set containing z then the set,

$$A = \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} p^n(U)$$

contains at most two points and is contained in E(p). Intuitively, the union of the forward orbits of points in the set U will contain every point in  $\hat{\mathbb{C}}$  apart from the exceptional points, of which there are at most two.

**Proof.** If this set had more than two points then  $\{p^n : n \ge 1\}$  would be equicontinuous on U by previous theorem. Then  $U \subseteq F(p)$  which is a contradiction since  $z \in U$  and  $z \in J(p)$ . Say there were a point  $w \in A$  which is not in E(p). Then [w] is infinite by definition of E(p)which implies the set of inverse iterates  $O^{-}(w) = \bigcup p^{-n}(w)$  is infinite. Since A has at most two points, some inverse iterate of w must be in  $\bigcup_{n=1}^{\infty} p^n(U)$  which means  $w \in \bigcup_{n=1}^{\infty} p^n(U)$ . This contradicts our choice of

contradicts our choice of w.

At last we have an obvious interesting conclusion from the previous theorem-heavy section. We've covered a fair bit of ground in terms of complex analysis to get here having seen the theorems of Ascoli-Arzela, Montel and Picard. This corollary also demonstrates a nice dichotomy between the orbits of certain points and the exceptional set. This connection will become even more clear in subsequent theorems but first we need another quite elementary result.

#### **Proposition 7.4.3.** If p is a polynomial function with deg $(p) \ge 2$ , then J(p) is infinite.

**Proof.** Firstly, we need to show that  $J(p) \neq \emptyset$ . Assume the contrary, that  $J(p) = \emptyset$ . Then  $\{p^n : n \geq 1\}$  is equicontinuous on  $\hat{\mathbb{C}}$  meaning  $F(p) = \hat{\mathbb{C}}$ . We know  $\infty$  is a superattracting fixed point of p so the iterates tend to infinity in its basin of attraction. By Vitali's (see [3, p 56]) theorem  $p^n \to \infty$  uniformly on compact subsets of  $\hat{\mathbb{C}}$ . And since  $F(p) = \hat{\mathbb{C}}$  is compact,  $p^n \to \infty$ on F(p). This is a contradiction since p must also have a finite fixed point. Thus  $J(p) \neq \emptyset$ .

Now we know there exists some point  $z \in J(p)$ . Since J(p) is completely invariant, if it is finite then z must be an exceptional point as [z] would be finite. This is a contradiction because we know that the exceptional points are contained in the Fatou set. Therefore J(p) must be infinite.

#### 7.4. INVERSE ITERATION METHOD

Now we'll introduce a new concept which will all but complete our analysis of the Julia set. The following definition was coined by Georg Cantor.

**Definition 7.4.4** (Derived Set). For a subset S of a topological space X, the *derived set* S' (of S) is the set of limit points of S. We'll use this to simplify the next definition.

**Definition 7.4.5** (Perfect Set). A subset S of a topological space X is called a *perfect set* if it is the set of limit points (accumulation points) of S. In other words, every point of S is a limit point of a sequence in S. Equivalently, a subset S of a topological space X is called a *perfect set* if it is equal to its derived set, i.e. if S = S'.

From this we can say that a perfect set is a closed set with no isolated points.

**Theorem 7.4.6.** If p is a polynomial function with deg  $(p) \ge 2$ , then J(p) is a perfect set.

**Proof.** We know that  $J(p)' \subseteq J(p)$  because J(p) is closed. Because J(p) is an infinite closed subset of a compact space, it is also compact and therefore J(p)' is non-empty. We claim that J(p)' is invariant. Because p is surjective we need only show backwards invariance  $p^{-1}(J(p)') = J(p)'$ . Firstly, for  $z \in J(p)'$  there exists a sequence  $(z_n)_{n=1}^{\infty} \subseteq J(p)$  such that  $z_n \to z$  as  $n \to \infty$ . Then by continuity of p,

$$p(z_n) \to p(z) \implies p(z) \in J(p)'$$
$$\implies p(J(p)') \subseteq J(p)'$$
$$\implies J(p)' \subseteq p^{-1}(J(p)')$$

since  $p(z_n)$  is in J(p) due to invariance. Secondly, fix  $z \in p^{-1}(J(p)')$  and let w = p(z). If u is an open neighbourhood around z then, p(U) is an open set around p(z) = w. We know  $w \in J(p)'$  and this implies  $p(U) \cap J(p) \neq \emptyset$  since w is in there. But also we have,

$$p^{-1}(p(U) \cap J(p)) = U \cap p^{-1}(J(p)) = U \cap J \neq \emptyset$$

Hence,  $z \in J(p)$  and so  $p^{-1}(J(p)') = J(p)'$ , which when combined with the first part means  $p^{-1}(J(p)) = J(p)$  and thus J(p)' is invariant. Now we have that J(p)' is a closed, invariant set distinct from F(p) and so because J(p) is already minimal,  $J(p) \subseteq J(p)'$ . The reverse inclusion was the first claim of this proof. Thus J(p) = J(p)' and J(p) is perfect.

This shows that every point of J is a limit point or accumulation point. The following theorem formalises a remarkable application of this development.

**Theorem 7.4.7.** If p is a polynomial with deg  $(p) \ge 2$ , and  $z \notin E(p)$ , then J(p) is contained in the closure of  $O^{-}(z)$ . Furthermore, if  $z \in J(p)$ , then J(p) is the closure of  $O^{-}(z)$ .

**Proof.** Say  $z \notin E(p)$ . For  $w \in J(p)$  and an open set U containing w, we must have  $z \in p^n(U)$  for some integer  $n \ge 1$ . Then there exists  $\zeta \in U$  such that  $p^n(\zeta) = z$ . Since  $p^{-n}(z) \in O^-(z)$  we now have that U contains a point in  $O^-(z)$ . Therefore J is contained in the closure of  $O^-(z)$ . If we furthermore assume that  $z \in J(p)$ , then by invariance of J(p),  $p^{-n}(z) \subseteq J$ . Since J is

closed, it must contain the closure of  $O^{-}(z)$ . If we combine this with the first part, then we see that J is equal to the closure of  $O^{-}(z)$ .

This theorem shows that the limit of the backwards orbit of almost any point in  $\hat{\mathbb{C}}$  gives a point in the Julia set. The inverse iterates of any non-exceptional point are infinite and converge to a point on the boundary of the Julia set. This simple observation yields a fairly naïve algorithm for computing Julia sets.

J(p) can be computed explicitly using a method called the inverse iteration method (IIM). This is done by taking the inverse of the given polynomial function and iterating this inverse. For an informal discussion, consider the earlier case where  $p(z) = z^2 + c$ . By iterating the inverse function  $p^{-1}(z) = \sqrt{z-c}$  one will eventually get a point on the border of the Julia set. Obviously with the square root, there usually exists two inverse iterates for each point. For example note that in this case,

$$\begin{aligned} (p \circ p^{-1})(z) &= p(p^{-1}(z)) = p(\sqrt{z-c}) = z - c + c = z \\ (p^{-1} \circ p)(z) &= p^{-1}(z^2 + c) = \sqrt{z^2 + c - c} = \sqrt{z^2} = \pm z \end{aligned}$$

so the number of inverse iterates doubles at every iteration of  $p^{-1}$ . The algorithm normally is pre-programmed to pick one or the other of these values. As a primitive example take c = 0, then  $p(z) = z^2$  and the inverse is  $p^{-1}(z) = \sqrt{z}$ . If we take the positive root every time and start at the point  $2 \in F(p)$  we get,

$$f(2) = 2^{\frac{1}{2}} = 1.414$$
  

$$f^{2}(2) = 2^{\frac{1}{4}} = 1.068$$
  

$$f^{3}(2) = 2^{\frac{1}{8}} = 1.033$$

where  $f^n(2) \to 1$  as  $n \to \infty$  which is a point in J(p).

Although this method is fairly naïve and costly in computing terms, most of the more refined algorithms used nowadays for generating pictures of Julia sets are based on the analysis in the previous chapter and the algorithm presented above. The revival of the area of Complex Dynamics in the 1980's was aided massively by the development of computer technology to the point where Julia sets for more complicated examples could be computed and presented in an elegant fashion to those unfamiliar with the subject.

## Chapter 8

## Components of the Fatou Set

## 8.1 Classification

**Definition 8.1.1.** A component U of the Fatou set F(R) is said to be:

- *periodic* if for some integer  $n \ge 1$ ,  $R^n(U) = U$ ,
- eventually periodic if for some integer  $m \ge 1$ ,  $R^m(U)$  is periodic
- wandering if all the sets  $R^n(U)$  for  $n \ge 1$  are pairwise disjoint.

A component of F(R) which is wandering is also called a wandering domain. The following theorem, due to Sullivan, shows that no such domains actually exist.

**Theorem 8.1.2** (Sullivan). Every component of the Fatou set F(R) of a rational map is eventually periodic (or periodic).

This theorem which was proved by Dennis Sullivan in 1985, is more generally known as the "no wandering domains" theorem. By eventually periodic we mean pre-periodic as in the definition given already. There are examples of entire functions in particular which have wandering domains as Baker's proved in his paper [8] (in fact he proved a conjecture of his own from some years previous). Rather than prove this quite lengthy theorem rigorously, we'll simply provide a sketch of the proof. I'll warn the reader that a lot of the specifics are omitted here and for a full version of the proof consult [3, Ch 8].

**Proof.**[(Sketch)] The first simplification relies on a theorem by Baker which says that the existence of a wandering domain implies the existence of a simply connected wandering domain. Specifically, if U is a wandering domain then  $\mathbb{R}^n(U)$  is simply connected for sufficiently large n [9].

From the Cauchy-Riemann equations we know that a function f is analytic if  $\frac{\partial f}{\partial \overline{z}} = 0$ . From this we introduce the Beltrami equation  $\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$  where  $\mu$  is a complex-valued function known as the Beltrami coefficient. If  $\mu = 0$  then a solution of the Beltrami equation is analytic.  $\mu$ 

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is a measure of how close a solution to  $\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$  is to being conformal. We say that f is quasiconformal with deviation  $\mu$ . While this only sets the background, the remainder of the proof relies heavily on manipulation of this Beltrami coefficient.

**Proposition 8.1.3.** If the Fatou set F(R) of a rational map consists of a finite number of components then they are all periodic.

**Proof.** Say there are finitely many components and let  $U_0$  be one of them. The claim is that  $U_0$  must be periodic. From Sullivan's theorem we know that the only other possibility is that  $U_0$  is pre-periodic so lets assume that and find a contradiction. Any rational function R maps the Fatou set F(R) onto F(R) since otherwise, R(F(R)) would be strictly contained in F(R) and there would be a component  $U_i \subseteq F((R))$  with  $R(U_i) \notin R(F(R))$  which contradicts invariance of the Fatou set. If we're assuming that there are a finite number of components  $U_i$  then this map will also be injective.

 $U_0$  pre-periodic means that  $R^n(U_0) \neq U_0$  for all  $n \geq 1$  and  $R^{n+j}(U_0) = R^j(U_0)$  for some  $n, j \geq 1$ . In other words some iterate  $R^j(U_0)$  is periodic. Now,

$$R^{n+j}(U_0) = R^j(U_0)$$
 and  $R^{n+j}(U_0) = R^j(R^n(U_0))$ 

Since  $R^n(U_0) \neq U_0$ , these are two distinct components of F(R) mapped to the same component by  $R^j$  which contradicts injectivity of R. Thus  $R^n(U_0) = U_0$  for some  $n \geq 1$  and therefore  $U_0$ is periodic.

**Theorem 8.1.4.** The number of components of the Fatou set can be  $0, 1, 2, \infty$  and all cases occur.

**Proof.** Assume first that the number of components is finite. Then from previous proposition we know that the components are periodic. Furthermore, we have that there exists N such that  $R^N(U) = U$  for every component U. So every component is invariant under  $R^N$ . By theorem 7.2.7, there are at most 2 of these components [2, p 70].

Here are examples (most of which we've seen before) of the cases where the Fatou set has  $0, 1, 2, \infty$  components.

**Example 8.1.5.** The Lattés function  $l(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}$  has Julia set  $J(l) = \hat{\mathbb{C}}$  and Fatou set  $\emptyset$  and therefore has 0 components. There is a theorem which states that for a rational function R, if every critical point of R is pre-periodic then  $J(R) = \hat{\mathbb{C}}$  [3, p 75]. The above is probably the most famous example and was discovered by Samuel Lattés. As a simpler case take John Guckenheimer's example  $R(z) = \frac{(z-2)^2}{z^2}$ . It is easy to check that this map has three critical points at  $0, 2, \infty$ . However, under this map  $2 \mapsto 0 \mapsto \infty \mapsto 1 \mapsto 1 \mapsto \ldots$  which means all three points are pre-periodic and thus R has Fatou set  $F(R) = \emptyset$ . The only justification worth

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making is that  $\infty \to 1$ ; the rest are obvious. To see this let,

$$f(z) = \frac{z^2 - 4z + 4}{z^2} = \frac{1 - \frac{4}{z} + \frac{4}{z^2}}{1}$$

This gives  $\lim_{z \to \infty} f(z) = 1.$ 

**Example 8.1.6.** The Fatou set of the map  $p(z) = z^2 + c$  where |c| > 2 has one connected component. In fact, this is true for all  $c \notin \mathcal{M}$  where  $\mathcal{M}$  is the Mandelbrot set.

**Example 8.1.7.** The Fatou set of the squaring map  $p(z) = z^2$  has 2 components namely  $\{z \in \hat{\mathbb{C}} : |z| > 1\}$  and  $\{z \in \hat{\mathbb{C}} : |z| < 1\}$ .

**Example 8.1.8.** The map  $p(z) = z^2 - 1$  has a Fatou set with an infinite number of components.

We know from Sullivan's theorem that every component of the Fatou set is periodic or eventually periodic. In order to classify these components we shall now state several definitions and combine them in a subsequent theorem.

**Definition 8.1.9** (Periodic Components). A periodic component U (with period n) of the Fatou set F(R) is called:

- *parabolic* if there exists a neutral fixed point  $z \in \partial U$  for  $\mathbb{R}^n$  such that all points in U converge to z.
- a Herman ring if it is conjugate to an irrational rotation of some annulus  $\{z \in \hat{\mathbb{C}} : 0 < r < |z| < 1\}$  onto itself.
- a Siegel disk if it is simply connected and R<sup>n</sup> is conjugate to an irrational rotation of the disk {z ∈ Ĉ : |z| < 1} onto itself.</li>

"Conjugate to an irrational rotation" means there exists a conjugation map  $\varphi$  such that  $(\varphi \circ f \circ \varphi^{-1})(z) = e^{i\theta}z$  where  $\theta$  is irrational.

**Theorem 8.1.10.** Suppose U is a periodic component of the Fatou set F(R). Then exactly one of the following holds:

- U contains an attracting fixed point,
- U is parabolic,
- U is a Siegel disk,
- U is a Herman ring.

For the proof see [2, Sec 4.2] or [3, Ch 7]. The existence of Siegel disks and Herman Rings was unsurprisingly proved by Carl Ludwig Siegel and Michael Herman respectively. Their existence was one of the main problems which Fatou and Julia encountered in their analysis of Complex Dynamical systems.

### 8.1. CLASSIFICATION CHAPTER 8. COMPONENTS OF THE FATOU SET

We've dealt a lot of the main aspects of dynamics in this paper. Specifically, Sullivan's "No Wandering Domains" theorem, Koenig's theorem and the Inverse Iteration Method (IIM), are three great eclectic examples of how the field has been developed over the past century. Firstly Sullivan proved that rational functions can't have wandering domains. This initially was a strong reason for the ailing interest in complex dynamics post Fatou and Julia, who hadn't managed to prove this conjecture.

Koenig's theorem was based largely on the work of Ernst Schröder ten years previous [7, p 10], while the IIM emphasises how modern computational power revived the subject in the 1980's.

Having said all this, for any avid readers of the field there are a few areas which deserve further investigation. In particular the paper by Jean-Christophe Yoccoz [10] (which won a fields medal in 1994) would be very interesting to analyse in depth as his results were some of the most important recent developments in the area. Specifically, his use of the Wolff-Denjoy theorem (which is actually a collection of separate results from Arnaud Denjoy and J. Wolff) seems to have a particular elegance; one might say a mathematical "Je ne sais quoi" - quite apt given that he was a Frenchman. Also, the work of Douady and Hubbard which we've mentioned in connection with the Mandelbrot set is a bounty of mathematics waiting to be discovered. Again, their conclusions were so powerful that they wouldn't be lost on one who would be less familiar with complex dynamics.

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