Question 9

Question 0.1 *What is meant by saying that a sequence $z_1, z_2, z_3, \ldots$ of complex number is a Cauchy Sequnce.*

A sequence $z_1, z_2, z_3, \ldots$ of complex numbers is a Cauchy Sequence IFF given any $\epsilon > 0$ $\exists$ some $N \in \mathcal{N}$ s.t $|z_m - z_n| < \epsilon \; \forall \; m, n \geq N$.

Question 0.2 *Prove that every convergent sequence of compex number is a Cauchy Sequence.*

If the sequence $z_1, z_2, z_3, \ldots$ is convergent then $\exists$ some $N \in \mathcal{N}$ s.t given any $\epsilon > 0$ $|z_m - l| < \frac{1}{2}\epsilon$ when $m \geq N$ and $|z_n - l| < \frac{1}{2}\epsilon$ when $n \geq N$ and where $l = \lim_{n \to +\infty} z_n$

$$\implies |z_m - z_n| = |(z_m - l) - (z_n - l)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

$\implies |z_m - z_n| < \epsilon$ which means that z_n is a Caucy Sequence. ▮

Question 0.3 *Prove that every Cauchy Sequence of complex number is bounded.*

Let the sequence $z_1, z_2, z_3, \ldots$ be a Cauchy Sequence. $\implies \exists$ some $N \in \mathcal{N}$ s.t. $|z_m - z_n| < 1 \; \forall \; m, n \geq N$
In Particular $|z_n| \leq |z_N| + 1$ when $n \geq N$
$\implies |z_n| \leq \; R$ where $R = max(|z_1|, |z_2|, \ldots, |z_{N-1}|, \ldots, |z_N| + 1)$
Therefore the sequence is bounded as required. ▮

Question 0.4 *Prove that every Cauchy sequence of complex numbers is convergent.*

Cauchy Sequences are bounded therefore have convergent subsequences by the Bolzano Weierstrass theorem.

Let $z_{n_1}, z_{n_2}, \ldots$ be one such subsequence. Therefore $\lim_{n \to +\infty} z_{n_j} = l$
We claim that $\lim_{n \to +\infty} z_n = l$
Let $\epsilon > 0$ be given therefore $\exists$ some $N \in \mathcal{N}$ s.t. $|z_m - z_n| < \frac{1}{2}\epsilon$ when $m, n \geq N$.
Let j be chosen s.t. $n_j \geq N$ and $|z_{n_j} - z_n| < \frac{1}{2}\epsilon$

$$|z_n - l| = |z_n - z_{n_j}| + |z_{n_j} - l| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \; when \; n \geq N$$

$$\implies |z_n - l| < \epsilon$$

Therefore every Cauchy Sequence is convergent. ▮