

Question 8

Question 0.1 *What is meant by saying that an infinite sequence z_1, z_2, z_3, \dots of complex numbers is convergent.*

An infinite sequence of complex numbers z_1, z_2, z_3, \dots is said to converge to some $l \in \mathcal{C}$ if given any $\epsilon > 0 \exists$ some $N \in \mathcal{N}$ s.t. $|z_n - l| < \epsilon \forall n \geq N$.

Question 0.2 *Suppose that $z_n = x_n + iy_n$ for all $n \in \mathcal{N}$ and $i = \sqrt{-1}$. Suppose the sequence x_1, x_2, x_3, \dots converges to λ and that the sequence y_1, y_2, y_3, \dots converges to μ where $\lambda, \mu \in \mathcal{R}$. Prove that the sequence z_1, z_2, z_3, \dots converges to the complex number $\lambda + i\mu$*

Suppose that $z_n \rightarrow l$ as $n \rightarrow +\infty$ where $l = \lambda + i\mu$. Then given any $\epsilon > 0 \exists$ some $N \in \mathcal{N}$ s.t. $|z_n - l| < \epsilon \forall n \geq N$.

$\implies |x_n - \lambda| < \epsilon$ & $|y_n - \mu| < \epsilon \forall n \geq N$

Therefore $x_n \rightarrow \lambda$ and $y_n \rightarrow \mu$ as $n \rightarrow +\infty$.

Conversely suppose that $x_n \rightarrow \lambda$ and $y_n \rightarrow \mu$ as $n \rightarrow +\infty$ then given any $\epsilon > 0 \exists$ some $N_1, N_2 \in \mathcal{N}$ s.t. $|x_n - \lambda| < \frac{1}{\sqrt{2}}\epsilon$ when $n \geq N_1$ and $|y_n - \mu| < \frac{1}{\sqrt{2}}\epsilon$ when $n \geq N_2$.

Let $N = \max(N_1, N_2)$ therefore when $n \geq N$ then

$$|z_n - l|^2 = |x_n - \lambda|^2 + |y_n - \mu|^2 \leq \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^2 + \epsilon^2$$

$$\implies |z_n - l| < \epsilon \text{ as required} \quad \blacksquare$$

Question 0.3 *Let (z_n) and (w_n) be convergent infinite sequences of complex numbers. Prove that the infinite sequence $(z_n + w_n)$ is convergent.*

Suppose $z_n \rightarrow l_1$ as $n \rightarrow +\infty$ and $w_n \rightarrow l_2$ as $n \rightarrow +\infty$. Where $z_n = x_{n_1} + iy_{n_1}$ and $w_n = x_{n_2} + iy_{n_2}$. And $l_1 = \lambda_1 + i\mu_1$ and $l_2 = \lambda_2 + i\mu_2$.

Then given any $\epsilon > 0 \exists$ some $N_1, N_2 \in \mathcal{N}$ s.t. $|z_n - l_1| < \epsilon$ when $n \geq N_1$ and $|w_n - l_2| < \epsilon$ when $n \geq N_2$

$$\implies |x_{n_1} - \lambda_1| < \epsilon \text{ \& } |y_{n_1} - \mu_1| < \epsilon \text{ when } n \geq N_1$$

$$\implies |x_{n_2} - \lambda_2| < \epsilon \text{ \& } |y_{n_2} - \mu_2| < \epsilon \text{ when } n \geq N_2$$

$$\implies x_{n_1} \rightarrow \lambda_1, y_{n_1} \rightarrow \mu_1, x_{n_2} \rightarrow \lambda_2, y_{n_2} \rightarrow \mu_2 \text{ all as } n \rightarrow +\infty$$

Conversely suppose that $x_{n_1} \rightarrow \lambda_1, y_{n_1} \rightarrow \mu_1, x_{n_2} \rightarrow \lambda_2, y_{n_2} \rightarrow \mu_2$ all as $n \rightarrow +\infty$.

Let $\epsilon > 0$ be given then $\exists N_1, N_2, N_3, N_4 \in \mathcal{N}$ s.t.

$|x_{n_1} - \lambda_1| < \frac{1}{4}\epsilon$ when $n \geq N_1$, $|x_{n_2} - \lambda_2| < \frac{1}{4}\epsilon$ when $n \geq N_2$,

$|y_{n_1} - \mu_1| < \frac{1}{4}\epsilon$ when $n \geq N_3$ and $|y_{n_2} - \mu_2| < \frac{1}{4}\epsilon$ when $n \geq N_4$. Let

$N = \max(N_1, N_2, N_3, N_4)$ then

$$|z_n - l_1| + |w_n - l_2| = |x_{n_1} - \lambda_1| + |y_{n_1} - \mu_1| + |x_{n_2} - \lambda_2| + |y_{n_2} - \mu_2| \leq \frac{1}{4}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \epsilon$$

$$|z_n - l_1| + |w_n - l_2| < \epsilon \implies (z_n + w_n) \text{ is convergent.} \quad \blacksquare$$

Question 0.4 What is meant by saying that an infinite series $\sum_{n=1}^{+\infty} s_n$ is convergent?

An infinite series $\sum_{n=1}^{+\infty} s_n$ is said to converge to some $s \in \mathcal{C}$ IFF given any $\epsilon > 0 \exists$ some $N \in \mathcal{N}$ s.t.

$$\left| \sum_{n=1}^m s_n - s \right| < \epsilon \quad \forall m \geq N$$

Question 0.5 Test the following infinite series for convergence.

$$\sum_{n=1}^{+\infty} \frac{\sin(n) - 4}{\sqrt{2n - 1}}$$

This diverges by the direct comparison test.

$$\sum_{n=1}^{+\infty} \frac{7\sin(n) - 4}{\sqrt{2n^3 - n^2}}$$

This converges by the direct comparison test.

$$\sum_{n=1}^{+\infty} \frac{z^n}{(2n)!}$$

This converges by the ratio test.

$$\sum_{n=1}^{+\infty} \frac{n!}{n^{n+3}}$$

This converges by the direct comparison test.