Question 7

Let $f_1, f_2, f_3, \dots$ be a sequence of continuous real-valued functions on an interval $[a,b]$ and let f be a real-valued function on $[a,b]$.

Question 0.1 *Define what is meant by saying that the functions f_n converge uniformly to f on the interval $[a,b]$ as $n \to \infty$*

The functions f_n are said to converge uniformly to some real-valued function f if given any $\epsilon > 0$ $\exists$ some $N \in \mathcal{N}$ s.t. $|f_n(x) - f(x)| < \epsilon \; \forall \; n \geq N$ and $\forall \; x \in [a,b]$.

Question 0.2 *Suppose that the function f_n converges uniformly to f on $[a,b]$ as $n \to +\infty$. Prove that*

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$$

Let $\epsilon > 0$ be given. Choose some ϵ_0 s.t. $0 < \epsilon_0(b-a) < \epsilon$. Now

$$-\int_a^b |f_n(t) - f(t)|dt \leq \int_a^b f_n(t)dt - \int_a^b f(t)dt \leq \int_a^b |f_n(t) - f(t)|dt$$

now because f_n converges uniformly to f on $[a,b]$ then

$$\left| \int_a^b f_n(t)dt - \int_a^b f(t)dt \right| \leq \int_a^b |f_n(t) - f(t)|dt \leq \epsilon_0(b-a) < \epsilon \; whenever \; n \geq N$$

The Result follows (because ϵ can be chosen arbitrarily small) ∎

Question 0.3 *Give an example of a sequence $f_1, f_2, f_3 \dots$ of real valued functions on an interval $[a,b]$ and a continuous real valued function f on $[a,b]$ s.t.*

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt \neq \int_a^b f(t)dt$$

even though $\lim_{n\to+\infty} f_n(t) = f(t)$ *for all* $t \in [a,b]$