Question 6

Question 0.1 *Let f be a function that is $k-$times differentiable and whose k^{th} derrivative is continuous on some open interval containing $a, a+h$ where $a, h \in \mathcal{R}$ Using the Rule for integration by parts show that*

$$f(a+h) = f(s) + \sum_{n=1}^{n} \frac{t^n}{n!} f^{(n)}(s) + r_k(a,h)$$

$$Where\ r_k(a,h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt$$

Take the expression for $r_k(a,h)$ and work on it integrating by parts.

$$\int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt = \int_0^1 udv$$

Where $u = f^{(k)}(a+th)\ and\ du = hf^{(k+1)}(a+th)dt$
And $v = \frac{-1}{k}(1-t)^k\ and\ dv = (1-t)^{k-1}dt$
note that $\int udv = uv - \int vdu$ Then

$$\int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt = [f^{(k)}(a+th)\frac{-1}{k}(1-t)^k]_0^1 - \int_0^1 \frac{-1}{k}(1-t)^k h f^{(k+1)}(a+th)dt$$

$$= \frac{1}{k} f^{(k)}(a) + \frac{h}{k}\int_0^1 (1-t)^k f^{(k+1)}(a+th)dt\ (k>0)$$

$$\Longrightarrow r_k(a,h) = \frac{h^k}{(k-1)!}\left(\frac{1}{k} f^{(k)}(a) + \frac{h}{k}\int_0^1 (1-t)^k f^{(k+1)}(a+th)dt\right)$$

$$= \frac{h^k}{k!} f^{(k)}(a) + r_{k+1}(a,h)$$

$$\Longrightarrow\ r_1(a,h) = h\int_0^1 f'(a+th)dt = \int_0^1 \frac{d}{dt}(f(a+th))dt = [f(a+th)]_0^1$$

$$\Longrightarrow f(a+h) = f(a) + r_1(a,h)$$

The required formula follows exactly by induction on k ∎