Question 5.

Question 0.1 *Let $f : [a, b] \Longrightarrow \mathcal{R}$ be a continuous function defined on the interval $[a, b]$ and let*

$$F(x) = \int_a^x f(t)dt, (a \leq x \leq b)$$

Prove that $F('x) = f(x)$ for all x s.t.$(a < x < b)$

f is continuous $\Longrightarrow$ given any $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t. $|f(t) - f(x)| < \frac{1}{2}\epsilon$ $\forall$ $t, x \in [a, b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))dt$$

If $0 < |h| < 1$ and $x + h \in [a, b]$. Then

$$\left|\int_x^{x+h} (f(t) - f(x))dt\right| < \frac{1}{2}\epsilon|h|$$

$$\Longrightarrow \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{1}{2}\epsilon < \epsilon$$

$$\Longrightarrow F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad \blacksquare$$

Question 0.2 *Find the Derrivative of the function $g : \mathcal{R} \Longrightarrow \mathcal{R}$ defined by*

$$g(x) = \int_0^{x^4} t^2 e^{t^2}$$

Using the fundemental theorem of calculus from above we know that if $F(x) = \int_a^x f(t)dt$ then $F'(x) = f(x)$
Now in this case

$$F(x) = \int_0^{x^4} t^2 e^{t^2} = F(t^2 e^{t^2})$$

$$F'(x) = 4x^{11} e^{x^8}$$