Question 4.

Question 0.1 *Let* $f : [a,b] \Longrightarrow \mathcal{R}$ *be a continuous function defined on the interval* $[a,b]$ *and let*

$$F(x) = \int_a^x f(t)dt, (a \leq x \leq b)$$

Prove that $F('x) = f(x)$ *for all* x *s.t.*$(a < x < b)$

f is continuous $\Longrightarrow$ given any $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t. $|f(t) - f(x)| < \frac{1}{2}\epsilon$ $\forall$ $t, x \in [a,b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) = \frac{1}{h}\int_x^{x+h}(f(t) - f(x))dt$$

If $0 < |h| < 1$ and $x + h \in [a,b]$. Then

$$\left|\int_x^{x+h}(f(t) - f(x))dt\right| < \frac{1}{2}\epsilon|h|$$

$$\Longrightarrow \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{1}{2}\epsilon < \epsilon$$

$$\Longrightarrow F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x) \quad \blacksquare$$

Question 0.2 *Use the method for integration by parts to evaluate*

$$\int_0^s t^3 e^{-t}dt \; for \; any \; s > 0$$

note $\int udv = uv - \int vdu$
Let $u_1 = t^3 \Longrightarrow du_1 = 3t^2dt$ and let $v_1 = -e^{-t} \Longrightarrow dv_1 = e^{-t}dt$

$$\int_0^s t^3 e^{-t}dt = -e^{-t}t^3 - \int_0^s(-e^{-t})3t^2dt$$

Let $u_2 = 3t^2 \Longrightarrow du_2 = 6tdt$ and let $v_2 = -e^{-t} \Longrightarrow dv_2 = e^{-t}dt$

$$\int_0^s(-e^{-t})3t^2dt = -3t^2e^{-t} - \int_0^s(-e^{-t})6tdt$$

Let $u_3 = 6t \Longrightarrow du_3 = 6dt$ and let $v_3 = -e^{-t} \Longrightarrow dv_3 = e^{-t}dt$

$$\int_0^s -e^{-t})6tdt = -6te^{-t} + 6\int_0^s(-e^{-t})dt = -6te^{-t} - 6e^{-t}$$

$$\Longrightarrow \int_0^s(-e^{-t})3t^2dt = -3t^2e^{-t} - 6te^{-t} - 6e^{-t}$$

$$\Longrightarrow \int_0^s t^3e^{-t}dt = [-e^{-t}t^3 - 3t^2e^{-t} - 6te^{-t} - 6e^{-t}]_0^s$$

$$\Longrightarrow \int_0^s t^3e^{-t}dt = -e^{-t}(s^3 + 3s^2 + 6s + 6) + 6$$