Question 3

Let $f : [a, b] \Longrightarrow \mathcal{R}$ be a bounded function defined on the interval $[a, b]$

**Question 0.1** *Define the concept of a questionition of the interval* $[a, b]$*.*
*Give the definition of lower sum* $L(P, f)$ *& upper sum* $U(P, f)$ *of* $f$ *for questionition* $P$

A questionition $P$ is a set $\{x_1, x_2, x_3, \ldots x_n\}$ $all \in \mathcal{R}$ s.t. $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.
The *lower sum* $L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$
The *upper sum* $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$

$Where\ m_i = inf\{f(x) : x_{i-1} < x < x_i\}$ & $M_i = sup\{f(x) : x_{i-1} < x < x_i\}$

**Question 0.2** *Define the lower Riemann Integral* $\mathcal{L}\int_a^b f(t)dt$ *and the upper Riemann Integral* $\mathcal{U}\int$ *of* $f$ *on interval* $[a, b]$*. Define precisely what is meant by saying that the function* $f$ *is Riemann Integrable on* $[a, b]$*, and define the Riemann Integral of a Riemann Integrable function on* $[a, b]$*.*

$$\mathcal{L}\int_a^b f(t)dt = sup(L(P, f))$$

$$\mathcal{U}\int_a^b f(t)dt = inf(U(P, f))$$

A function $f$ on $[a, b]$ is Riemann integrable if $\mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt$.
The Riemann Integral of $f$ on $[a, b]$ is the common value of $\mathcal{L}\int_a^b f(t)dt$ and $\mathcal{U}\int_a^b f(t)dt$.

**Question 0.3** *Let* $f : [0, 1] \Longrightarrow \mathcal{R}$ *be defined by* $f(t) = 1 - t^2$*. Calculate* $L(P_n, f)$ *and* $U(P_n, f)$*, where* $P_n$ *denotes the questionition of* $[0, 1]$ *into* $n$ *subintervals of length* $\frac{1}{n}$*. ie.* $P = \{t_0, t_1, \ldots, t_n\}$ *where* $t_i = \frac{1}{n}$ *for* $i = 0, 1, \ldots, n$*. Hence show that*

$$\mathcal{L}\int_a^b f(t)dt = (U)\int_a^b f(t)dt = \frac{2}{3}$$

$f$ takes values between $1 - \frac{(i-1)^2}{n^2}$ and $1 - \frac{i^2}{n^2}$ and $1 - \frac{i^2}{n^2} \leq 1 - \frac{(i-1)^2}{n^2}$
Using the definitions from above.

$$m_i = 1 - \frac{i^2}{n^2}\ and\ M_i = 1 - \frac{(i-1)^2}{n^2}$$

$$Now\ L(P_n, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{i^2}{n^2})$$

$$And\ U(P_n, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{(i-1)^2}{n^2})$$

$$\Longrightarrow L(P_n, f) = (1 - \frac{1}{n^3})\sum_{i=1}^{n} i^2 = \frac{1}{6n^2}(n+1)(2n+1) = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

$$\Longrightarrow U(P_n, f) = (1 - \frac{1}{n^3})\sum_{i=1}^{n-1} i^2 = \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$$

Then $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2} \leq \mathcal{L}\int_a^b f(t)dt \leq \mathcal{U}\int_a^b f(t)dt \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$
But The upper and lower Riemann Integrals do not depend on $n$ because it is constant.

$$\Longrightarrow \mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt = \frac{2}{3} \qquad \blacksquare$$