Question 2

Let $f : [a, b] \Longrightarrow \mathcal{R}$ be a bounded function defined on the interval $[a, b]$

**Question 0.1** *Define the concept of a partition of the interval $[a, b]$. Give the definition of lower sum $L(P, f)$ & upper sum $U(P, f)$ of $f$ for partition $P$*

A Partition $P$ is a set $\{x_1, x_2, x_3, \ldots x_n\}$ $all \in \mathcal{R}$ s.t. $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

The *lower sum* $L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$

The *upper sum* $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$

$$Where\ m_i = inf\{f(x) : x_{i-1} < x < x_i\}\ \&\ M_i = sup\{f(x) : x_{i-1} < x < x_i\}$$

**Question 0.2** *Define the lower Riemann Integral $\mathcal{L}\int_a^b f(t)dt$ and the upper Riemann Integral $\mathcal{U}\int$ of $f$ on interval $[a, b]$. Define precisely what is meant by saying that the function $f$ is Riemann Integrable on $[a, b]$, and define the Riemann Integral of a Riemann Integrable function on $[a, b]$.*

$$\mathcal{L}\int_a^b f(t)dt = sup(L(P, f))$$

$$\mathcal{U}\int_a^b f(t)dt = inf(U(P, f))$$

A function $f$ on $[a, b]$ is Riemann integrable if $\mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt$.

The Riemann Integral of $f$ on $[a, b]$ is the common value of $\mathcal{L}\int_a^b f(t)dt$ and $\mathcal{U}\int_a^b f(t)dt$.

**Question 0.3** *Explain why $L(P, f) \leq U(Q, f)$ for all partitions $P$ and $Q$ on $[a, b]$. and show that*

$$\mathcal{L}\int_a^b f(t)dt \leq \mathcal{U}\int_a^b f(t)dt$$

Let $R$ be a common refinement of $P$ and $Q$. Using the fact that

$$L(P, f) \leq\ L(R, f) \leq\ U(R, f) \leq\ U(P, f)$$

$$And\ L(Q, f) \leq\ L(R, f) \leq\ U(R, f) \leq\ U(Q, f)$$

It is clear from the above inequalities that $L(P, f) \leq\ U(Q, f)$

Taking the *sup* of the L.H.S. of this inequality we get

$$\mathcal{L}\int_a^b f(t)dt \leq\ U(Q, f)$$

and taking the $inf$ of the R.H.S. of this inequality we see that

$$\mathcal{L}\int_a^b f(t)dt \leq\ \mathcal{U}\int_a^b f(t)dt \qquad \blacksquare$$

**Question 0.4** *Let* $f : [a, b] \Longrightarrow \mathcal{R}$ *be a constant function with value* $c$*. What are the values of* $L(P, f)$ *&* $U(P, f)$ *for any partition* $P$ *on* $[a, b]$*.*

Now $f$ is a constant function. Therefore $L(P, f) = U(P, f)$. Using the definitions from above it is clear that $m_i = M_i$ and the value is $c$.

$$\Longrightarrow L(P, f) = U(P, f) = \sum_{i=1}^{n} (x_i - x_{i-1}).c \ = \ (b - a).c$$

**Question 0.5** *Let* $f : [0, 1] \Longrightarrow \mathcal{R}$ *be defined by*

$$f(t) = \{t \ if \ t \ is \ rational, 0 \ if \ t \ is \ irrational\}$$

*Is* $f$ *Riemann Integrable on* $[a, b]$*? Justify your answer.*

$f$ is not Riemann Integrable on $[a, b]$ because
Each interval $[x_i, x_{i-1}]$ contains both rational and irrational numbers

$$\Longrightarrow U(P, f) = t \ and \ L(P, f) = 0$$

$$\Longrightarrow \mathcal{L} \int_a^b f(t)dt \neq \mathcal{U} \int_a^b f(t)dt$$