

Question 16

Question 0.1 Prove that a sequence $x_1, x_2, x_3 \dots$ of points of \mathcal{R}^2 converges to some point p IFF, given any open set U which contains p , \exists some $N \in \mathcal{N}$ s.t. the point x_j belongs to $U \forall j$ s.t. $j \geq N$.

Suppose that the sequence $x_1, x_2, x_3 \dots$ has the property that given any open set U containing p exists some $N \in \mathcal{N}$ s.t. $x_j \in U$ when $j \geq N$.

Let $\epsilon > 0$ be given. The open ball $B(p, \epsilon)$ is an open set. $\implies \exists$ some $N \in \mathcal{N}$ s.t. $x_j \in B(p, \epsilon)$ when $j \geq N$.

$\implies |x_j - p| < \epsilon$ when $j \geq N$. Therefore the sequence converges.

Conversely. Suppose that the sequence converges to some p . Let U be an open set containing p . Then \exists some $\epsilon > 0$ s.t. the open ball $B(p, \epsilon)$ of radius ϵ around p is a subset of U .

$\implies \exists$ some $N \in \mathcal{N}$ s.t. $|x_j - p| < \epsilon$ when $j \geq N$ since the sequence converges to p .

$\implies x_j \in U$ when $j \geq N$ ■.

Question 0.2 Using part (a) or otherwise show that if F is a closed set in \mathcal{R}^2 and if x_1, x_2, x_3, \dots is an infinite series of points of \mathcal{R}^2 belonging to F which converges to some complex number p then $p \in F$.

The complement of F , $\mathcal{R}^2 \setminus F$ is open by definition. \implies Suppose $p \in \mathcal{R}^2 \setminus F$ then it follows that $x_j \in \mathcal{R}^2 \setminus F \forall j \geq N$.

This contradicts the fact that $x_j \in F \forall j \implies p \in F$ as required. ■